INDUSTRIAL mathematics

1961

I

M

TH

IN

EDI

RIC

THE

INDUSTRIAL MATHEMATICS

THE JOURNAL OF THE INDUSTRIAL MATHEMATICS SOCIETY

EDITED BY

RICHARD L. JUSTICE

VOLUME 11, PART 1

1961

THE INDUSTRIAL MATHEMATICS SOCIETY
DETROIT, MICHIGAN

Copyright 1960, by the Industrial Mathematics Society
100 Farnsworth Avenue, Detroit 2, Michigan
Library of Congress Catalog Card Number 53-39635

EDITORIAL STAFF

Managing Editor

R. L. Justice

Assistant Editors

R. Davies
D. Gazis

LITHOPRINTED IN THE UNITED STATES OF AMERICA BY
CUSHING - MALLOY, INC., ANN ARBOR, MICHIGAN, 1961

FOREWORD

The Industrial Mathematics Society is a professional organization whose objective it is to extend the understanding and application of mathematics in industry. Founded early in 1949, it marked the first organized attempt of any group to foster a closer relationship between mathematics and industry.

It promotes the cause of mathematics in industry through monthly meetings at which technical papers are presented, through lectures by nationally prominent scientists and engineers, through panel groups which concentrate in specialized fields of knowledge, and through publication of worthwhile papers in its annual volume.

Membership is open to engineers, scientists, and mathematicians who are concerned with the effective use of mathematics in industry. Annual dues are $5.00 per year including a subscription to the annual volume, *Industrial Mathematics*. Application forms and information may be obtained by writing to the Secretary, Industrial Mathematics Society, 100 Farnsworth, Detroit 2, Michigan.

EXECUTIVE OFFICERS
INDUSTRIAL MATHEMATICS SOCIETY

President, Joseph Rackle, Ford Motor Company, Engineering Staff
Vice President, Walter Kotoucek, Ford Motor Company, Engineering Staff
Treasurer, John Glover, Ford Motor Company, Transmission and Chassis Division
Secretary, William Buell, Ford Motor Company, Engineering Staff

CONTENTS

		Page
1.	A Composite Infinite Slider Bearing. . . . . Donald F. Hays	1
2.	A Two-Dimensional Theory of Stress Concentration Around Voids. . . . . Leonard G. Johnson	21
3.	Stability Study of a Fluid Flow Problem With Flow Reversal . . . . . Richard L. Justice	27
4.	Computers — Large or Small? . . . . . Robert K. Louden	37
5.	Parameters Needed for Stress Calculations on Cams . . . . . Robert Davies	43
6.	The Effect of Inclusions on the Fatigue Life of Steel . . . . . Leonard G. Johnson	57

UNIVERSITY OF UTAH LIBRARIES

1

A Composite Infinite Slider Bearing

By Donald F. Hays
Research Laboratories
General Motors Corporation

ABSTRACT

An analysis is made of an infinite slider bearing which is composed of three intersecting surfaces. Characteristics of such a composite slider are then compared with the corresponding characteristics of an infinite plane slider with the same film ratio.

NOTATIONS

A_n – Film height at surface intercepts
F – Friction force - F/L
$K_{1,n}, K_{2,n}$ – Integration constants
L – Slider length - L
P – Pressure - F/L^2
U – Relative velocity - L/T
W – Load Capacity - F/L
$\bar{X}$ – Center of pressure - L
f – Coefficient of friction
h_n – Film Thickness - L
h_o – Exit film thickness - L
h_i – Inlet film thickness - L
k – Film ratio = h_i/h_o
n – Subscript - 1, 2, 3 range
q – Flow Rate - L^2/T
u – Velocity - L/T
x_n, y – Coordinate axes
Δ – Amplitude ratio = δ/h_o
θ_n, y – Transformed coordinates
μ – Viscosity - $(FT)/L^2$
β_n – Angle of inclination
δ – Waviness amplitude - L

Note: Circumflexed symbols are dimensional ($\hat{}$)

INTRODUCTION

In the manufacture of bearings, care must be exercised in the preparation of the bearing surface. Two factors, surface roughness and surface waviness, must be taken into consideration. Surface roughness may be considered as a high frequency irregularity of the surface in contrast to waviness which is a low frequency phenomenon. Since waviness is sometimes found in ground bearing surfaces, it is of interest to investigate the effect of this irregularity on bearing performance. This is especially important when the amplitude of the waviness approaches an order of magnitude which is comparable to the minimum film thickness. The simplified model to be analyzed is an infinite composite slider bearing which is composed of three surfaces. These are orientated such that they form a linear wave form about a plane surface which is taken as the reference slider. For various amplitude ratios based on the outlet film height, the operational characteristics of the bearing are computed and are compared to the corresponding characteristics of the infinite plane slider. In all comparisons, the ratio of inlet film thickness to outlet film thickness is constant. The computed characteristics include the pressure distribution, load capacity, frictional force, coefficient of friction, center of pressure, and flow rate. The assumptions are that the fluid is incompressible, the viscosity is constant, and that the Reynolds' Number is small.

ANALYSIS

The infinite composite slider is shown in Fig. 1. It is composed of three sections of length L/3, the sections being inclined at the angles β_n. The entrance film thickness is h_i and the outlet film thickness is h_o. A separate coordinate system is used for each section and each section is solved with the appropriate boundary conditions so as to insure a continuous pressure function throughout the total length. The basic equation which must be satisfied for each section is the one-dimensional Reynolds' Equation.

$$\frac{d}{dX_n}\left(h_n^3 \frac{dP_n}{dX_n}\right) = -6\mu U \frac{dh_n}{dX_n} \tag{1}$$

The subscripts "n" have a range 1, 2, 3, and indicate the section being considered as measured from the trailing edge. The film thickness is given by the following form,

$$h_n = \hat{A}_n + \beta_n X_n \tag{2}$$

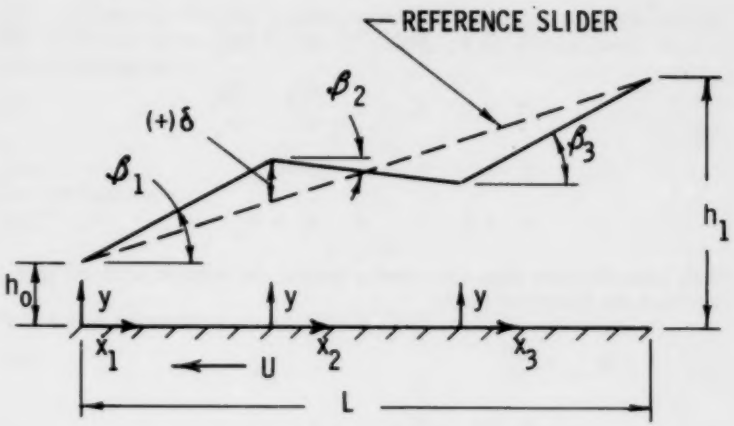


Figure 1. Diagram of Slider Bearings.

Equation (1) may be integrated twice to yield the pressure equations as shown.

$$P_n = \frac{6\mu U}{\beta_n(\hat{A}_n + \beta_n X_n)} - \frac{\hat{K}_{1,n}}{2\beta_n(\hat{A}_n + \beta_n X_n)^2} + \hat{K}_{2,n} \tag{3}$$

By the following transformation

$$\theta_n = \frac{3X_n}{L} \tag{4}$$

the nondimensional pressure may be written as

$$\frac{P_n h_0^2}{\mu U L} = \frac{54}{\beta_n(3A_n + \beta_n \theta_n)^2}\left[\frac{3A_n + \beta_n \theta_n}{3} - \frac{K_{1,n}}{2} + \frac{K_{2,n}\beta_n(3A_n + \beta_n \theta_n)^2}{9}\right] \tag{5}$$

These Equations (5) contain six arbitrary constants of integration which are evaluated such that the following six boundary conditions are satisfied:

$$\theta_1 = 0 \quad P_1 = 0 \tag{6}$$

$$\theta_3 = 1 \quad P_3 = 0 \tag{7}$$

$$\theta_1 = 1 \quad \theta_2 = 0 \quad \frac{dP_1}{d\theta_1} = \frac{dP_2}{d\theta_2} \tag{8}$$

$$\theta_2 = 1 \quad \theta_3 = 0 \quad \frac{dP_2}{d\theta_2} = \frac{dP_3}{d\theta_3} \tag{9}$$

$$\theta_1 = 1 \quad \theta_2 = 0 \quad P_1 = P_2 \tag{10}$$

$$\theta_2 = 1 \quad \theta_3 = 0 \quad P_2 = P_3 \tag{11}$$

After considerable algebraic manipulation, the constants of integration take the following form:

$$K_{1,1} = \frac{2M}{N} \tag{12}$$

$$K_{2,1} = \frac{1}{\beta_1}\left(\frac{M}{N} - 1\right) \tag{13}$$

$$K_{1,2} = \frac{2M}{N} \tag{14}$$

$$K_{2,2} = \frac{(\beta_1 - \beta_2)}{A_2 \beta_1 \beta_2}\left(\frac{M}{A_2 N} - 1\right) + \frac{1}{\beta_1}\left(\frac{M}{N} - 1\right) \tag{15}$$

$$K_{1,3} = \frac{2}{9}\left(\frac{9M}{N} - \frac{\beta_2^2}{A_3}\right) \tag{16}$$

$$K_{2,3} = \frac{3}{\beta_1 (3A_3 + \beta_1)^2}\left[\frac{3M}{N} - 3A_3 - \beta_1 - \frac{\beta_2^2}{3A_3}\right] \tag{17}$$

where M and N are given by the following expressions:

$$M = \sum_{n=1}^{3} \frac{A_{n+1}^3}{2}\left[\frac{9}{(3A_n + \beta_n)^2} - \frac{1}{A_{n+1}^2}\right]\left[\left\{\frac{27\, A_{2-n}^3}{\beta_n (3A_{n+1} + \beta_{n+1})^3}\right\}\left\{\frac{9}{(3A_{n+2} + \beta_{n+2})^2} - \frac{1}{A_{n+2}^2}\right\} - \frac{1}{\beta_{n+1}}\left\{\frac{9}{(3A_{n+1} + \beta_{n+1})^2} - \frac{1}{A_{n+1}^2}\right\}\right] \tag{18}$$

$$N = \sum_{n=1}^{3} \frac{27^{n-1} A_n}{\beta_n (3A_{n-1} + \beta_{n-1})^3 (3A_{n-2} + \beta_{n-2})^3}\left[\frac{9 A_n^2}{(3A_n + \beta_n)^2} - 1\right] \tag{19}$$

The expressions for A_n and B_n are determined by the geometry of the slider as shown in Fig. 1. By defining a nondimensional amplitude of waviness as

$$\Delta = \frac{\delta}{h_0} \tag{20}$$

and the film ratio as

$$K = \frac{h_1}{h_0} \tag{21}$$

then the expressions for A_n and B_n become

$$A_1 = 1 \tag{22}$$

$$A_2 = \frac{1}{3}(2 + K + 3\Delta) \tag{23}$$

$$A_3 = \frac{1}{3}(1 + 2K - 3\Delta) \tag{24}$$

$$\beta_1 = K - 1 + 3\Delta \tag{25}$$

$$\beta_2 = K - 1 - 6\Delta \tag{26}$$

$$\beta_3 = K - 1 + 3\Delta = \beta_1 \tag{27}$$

LOAD CAPACITY

The load capacity of the composite slider bearing is found by summing the load capacities of each section. Thus

$$W = \sum_{n=1}^{3} \int_0^{\frac{L}{3}} P_n \, dX_n \tag{28}$$

which when written in nondimensional form becomes

$$\frac{W h_0^2}{\mu U L^2} = 6 \sum_{n=1}^{3} \left[\left\{ \frac{1}{\beta_n^2} \ln\left(1 + \frac{\beta_n}{3A_n}\right) \right\} + \frac{1}{3} K_{2,n} - \frac{1}{2} \left\{ \frac{K_{1,n}}{A_n \beta_n (3A_n + \beta_n)} \right\} \right] \tag{29}$$

FRICTIONAL FORCE

The frictional drag experienced by the bearing as it moves with a velocity "U" is given by

$$F = - \sum_{n=1}^{3} \mu \int_{0}^{\frac{L}{3}} \frac{du_n}{dy} \, dX_n \tag{30}$$

where

$$u_n = \frac{1}{2\mu} \frac{dP_n}{dX_n} (y - h_n)y - U\left(1 - \frac{y}{h_n}\right) \tag{31}$$

In nondimensional form the friction force per unit width is

$$\frac{F}{\mu U} = \sum_{n=1}^{3} \frac{1}{\beta_n} \left[\frac{3\beta_n K_{1,n}}{A_n(3A_n + \beta_n)} - 4 \ln \left(1 + \frac{\beta_n}{3A_n}\right) \right] \tag{32}$$

CENTER OF PRESSURE

The center of pressure of the composite slider as measured from the exit section is found from the following expression:

$$\bar{X} = \frac{\sum_{n=1}^{3} \int_{0}^{\frac{L}{3}} P_n \left[X_n - \frac{(n-1)L}{3} \right] dX_n}{\sum_{n=1}^{3} \int_{0}^{\frac{L}{3}} P_n \, dX_n} \tag{33}$$

The denominator of Equation (33) is the load capacity as given by Equation (29) and the numerator of Equation (33) is given in nondimensional form as follows:

$$\frac{(\) h_0^2}{\mu U L^3} = 6 \sum_{n=1}^{3} \left[\frac{1}{3\beta_n^3} \left\{ \beta_n + (\{n-1\}\beta_n - 3A_n) \ln \left(1 + \frac{\beta_n}{3A_n}\right) \right\} \right.$$

$$- \frac{K_{1,n}}{2\beta_n^3} \left\{ \ln \left(1 + \frac{\beta_n}{3A_n}\right) - 1 + \frac{(n-1)\beta_n^2 + 9A_n^2}{3A_n(3A_n + \beta_n)} \right\}$$

$$\left. + \frac{(2n-1)}{18} K_{2,n} \right] \tag{34}$$

The nondimensional center of pressure as measured from the trailing edge is

$$\frac{\bar{X}}{L} = \frac{\text{Equation (34)}}{\text{Equation (29)}} \tag{35}$$

COEFFICIENT OF FRICTION

The coefficient of friction defined as the ratio of frictional resistance to normal load is given by

$$f\left(\frac{L}{h_0}\right)^2 = \frac{\text{Equation (32)}}{\text{Equation (29)}} \tag{36}$$

FLOW RATE

The flow rate per unit width is determined from

$$q = \frac{-h_0^3}{12\mu} \left.\frac{dP_1}{dX_1}\right|_{X_1 = 0} - \frac{U h_0}{2} \tag{37}$$

which reduces to the following nondimensional form

$$\frac{q}{U h_0} = \frac{-K_{1,1}}{2} \tag{38}$$

RESULTS

The characteristics of the composite slider bearing have been computed for film ratios up to 5.0 and for various amplitude ratios Δ. Since these amplitude ratios may be either positive or negative, two curve sets are given for each characteristic of the slider. It should be noted that for negative values of Δ there is a limiting value for k such that when

$$K < 1 - 3\Delta \tag{39}$$

there is an internal film thickness that is less than the outlet film thickness. In all curves, the ratio of the composite slider characteristics to the plane slider characteristics is made on the basis of the film ratio which is computed from the entrance and exit film thickness. For positive values of Δ the limiting case is for

$$K < \frac{3}{2}\Delta + 1 \qquad (40)$$

which also yields an internal film thickness that is less than the exit height.

In Figs. 2 and 3 are shown pressure curves for a slider with a film ratio of 3.0 and having various amplitude ratios Δ. The operational characteristics of a composite slider as compared with an ordinary plane slider are shown in Figs. 4 through 13. In Figs. 4 through 13, "Region I" indicates a minimum film thickness occurring at the exit section and "Region II" indicates an internal minimum film thickness. A symbol such as W(o) refers to the characteristic of the ordinary plane slider, i.e., Δ is zero.

CONCLUSIONS

The overall effect of waviness in the surface of an infinite plane slider is seen to be deleterious to the operation of the bearing. This effect becomes more pronounced as the amplitude ratio is increased for a value of the film ratio. Thus waviness present in the surface might not be detectable in its influence on the bearing operation at light loads, but could possibly contribute greatly to the reduction in load capacity at higher loads when the minimum film thickness decreases. The general effect of waviness is to aggravate the situation; higher loads decreasing the film thickness, a larger amplitude ratio then decreasing the load capacity of the unit which in turn would decrease the minimum film thickness. This cycle would continue until equilibrium was established.

For extremely thin films, the viscosity of the oil will continue to decrease due to the effect of temperature as the oil passes through the bearing. Since this analysis is based upon a constant average viscosity, some discretion must be exercised in the interpretation of the results.

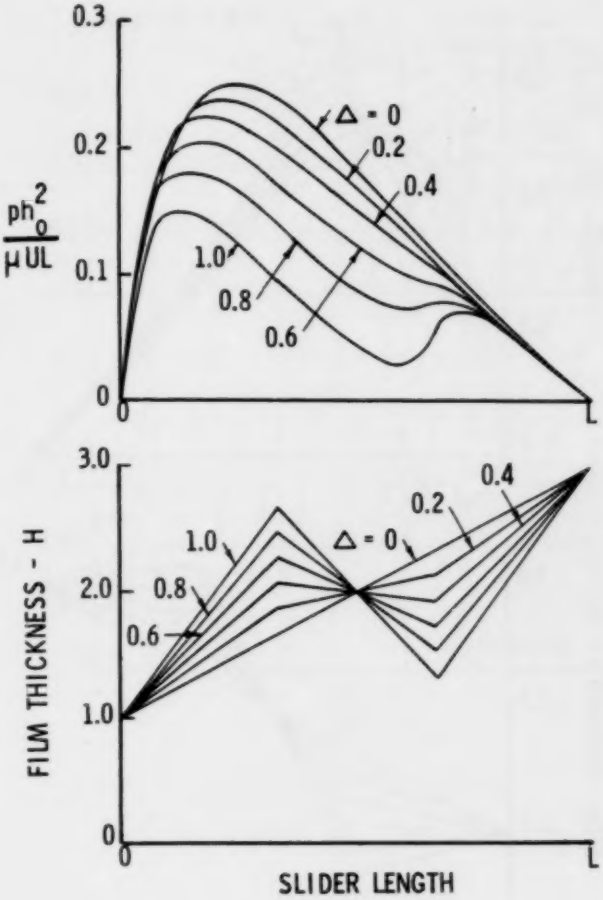


Figure 2. Pressure Distributions and Film Thickness Geometries for Positive Amplitude Ratios.

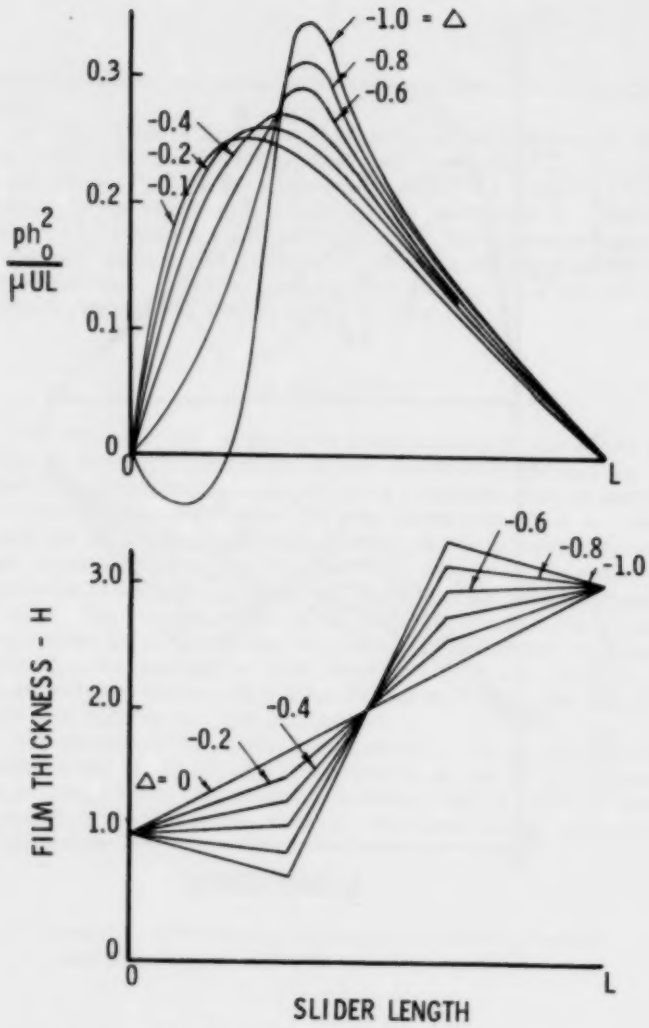


Figure 3. Pressure Distributions and Film Thickness Geometries for Negative Amplitude Ratios.

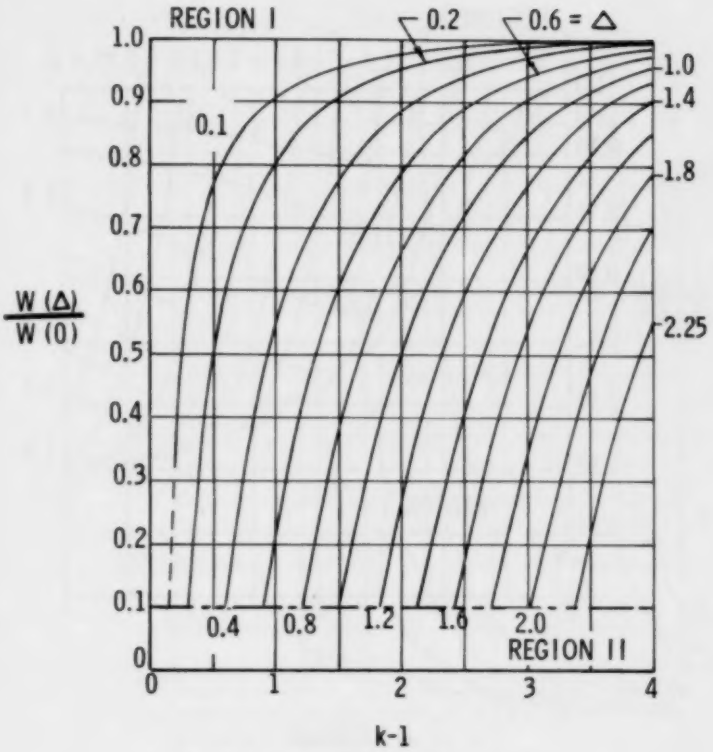


Figure 4. Load Capacity Ratio Curves for Positive Amplitude Ratios.

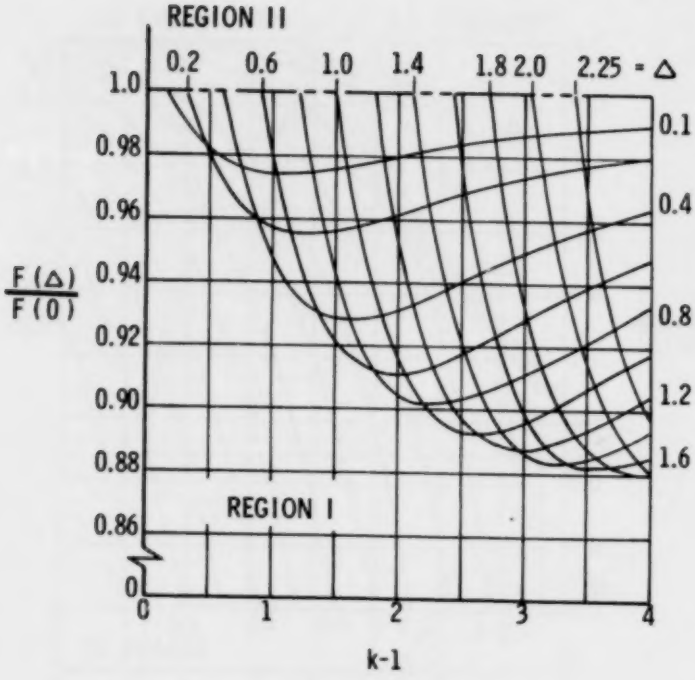


Figure 5. Friction Force Ratio Curves for Positive Amplitude Ratios.

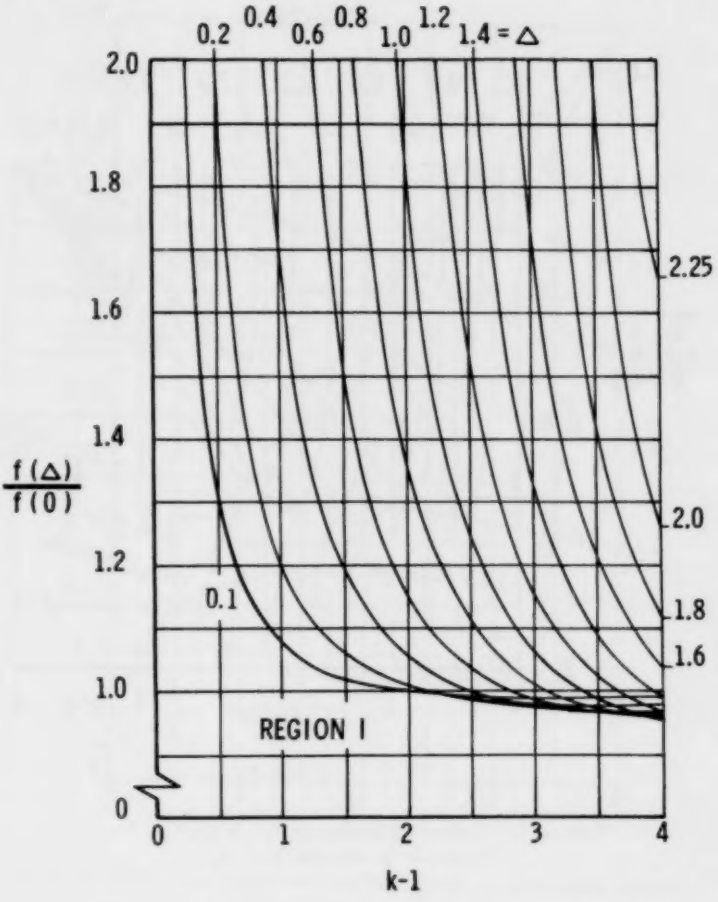


Figure 6. Coefficient of Friction Ratio Curves for Positive Amplitude Ratios.

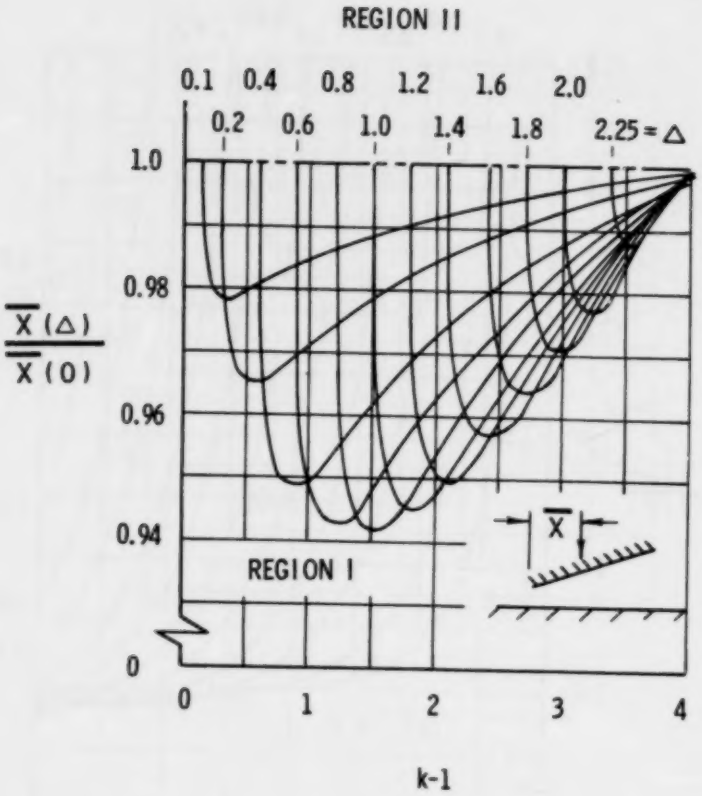


Figure 7. Center of Pressure Ratio Curves for Positive Amplitude Ratios.

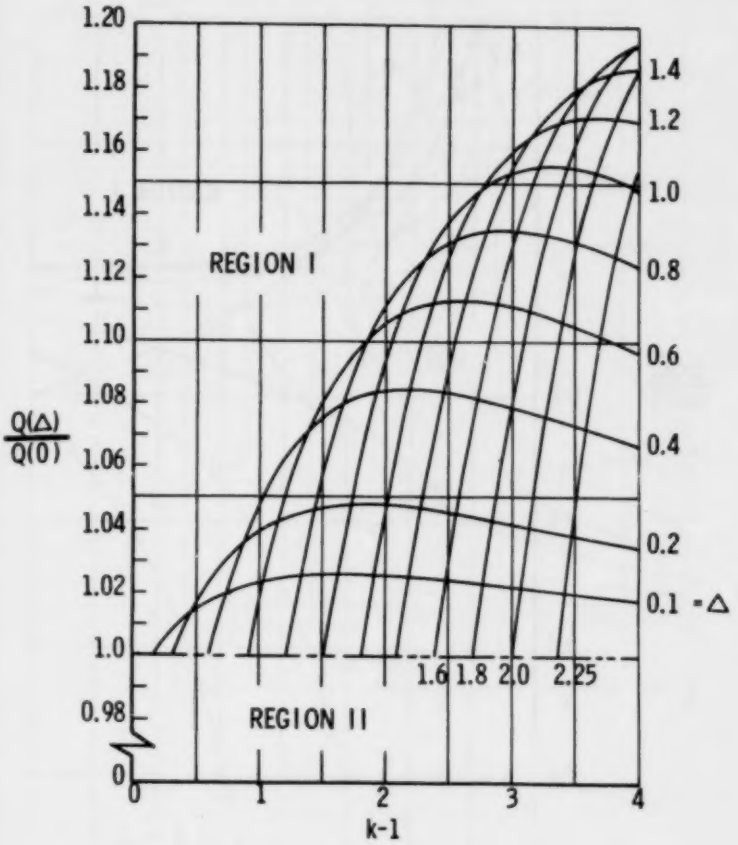


Figure 8. Flow Rate Ratio Curves for Positive Amplitude Ratios.

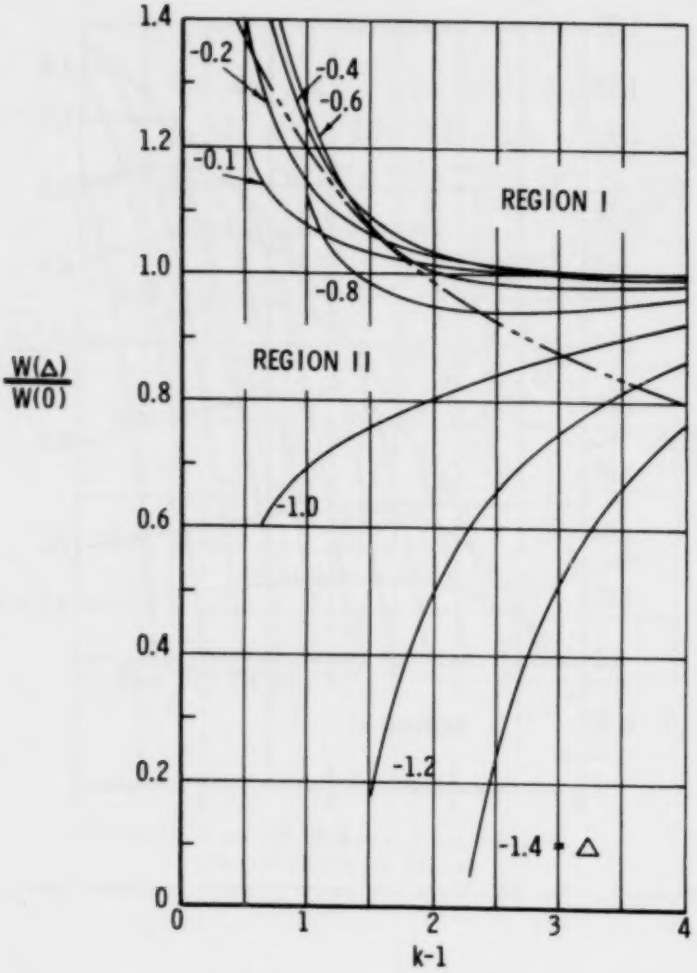


Figure 9. Load Capacity Ratio Curves for Negative Amplitude Ratios.

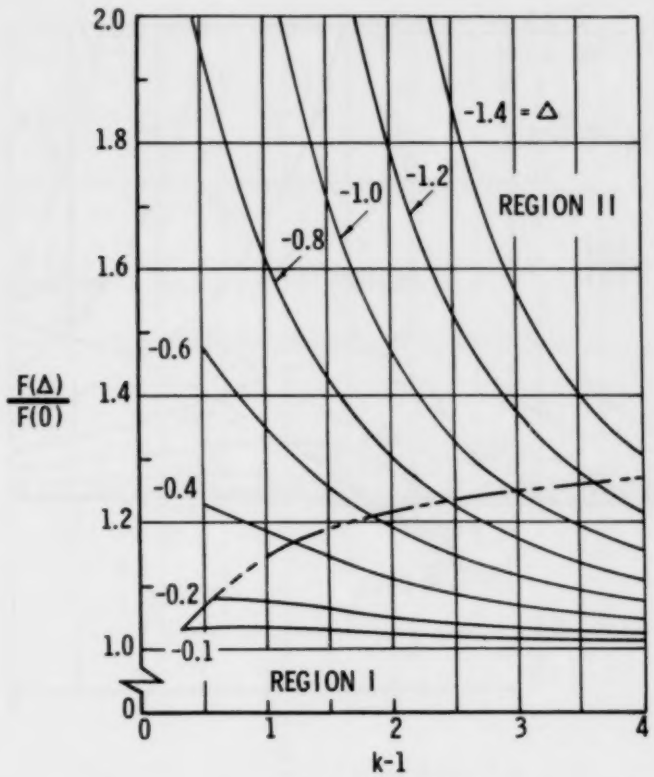


Figure 10. Friction Force Ratio Curves for Negative Amplitude Ratios.

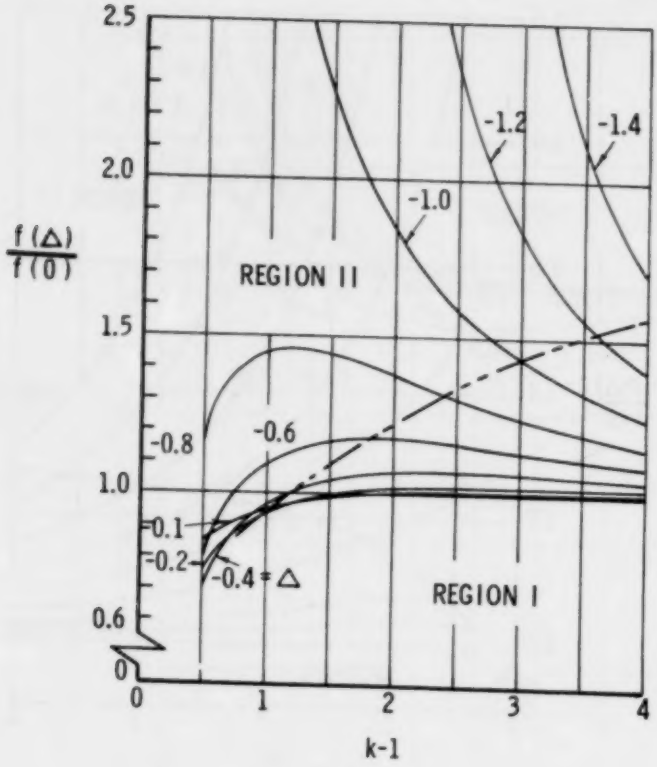


Figure 11. Coefficient of Friction Ratio Curves for Negative Amplitude Ratios.

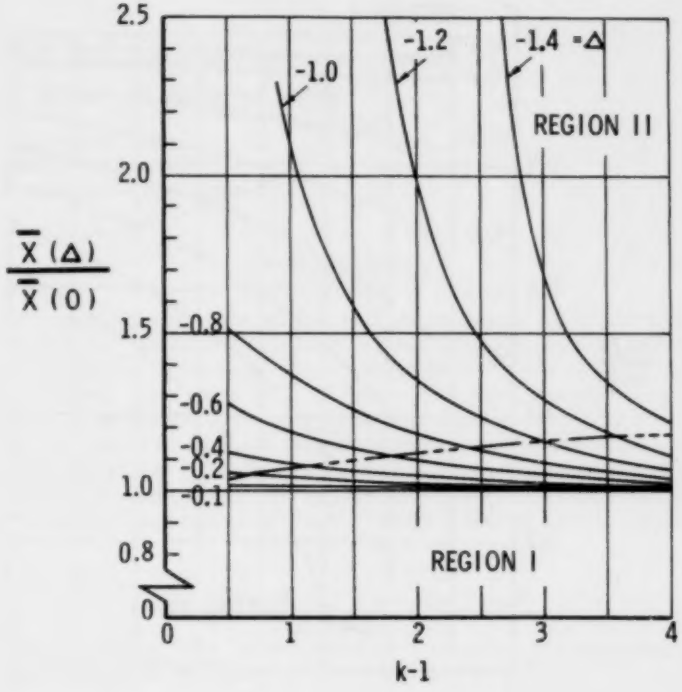


Figure 12. Center of Pressure Ratio Curves for Negative Amplitude Ratios.

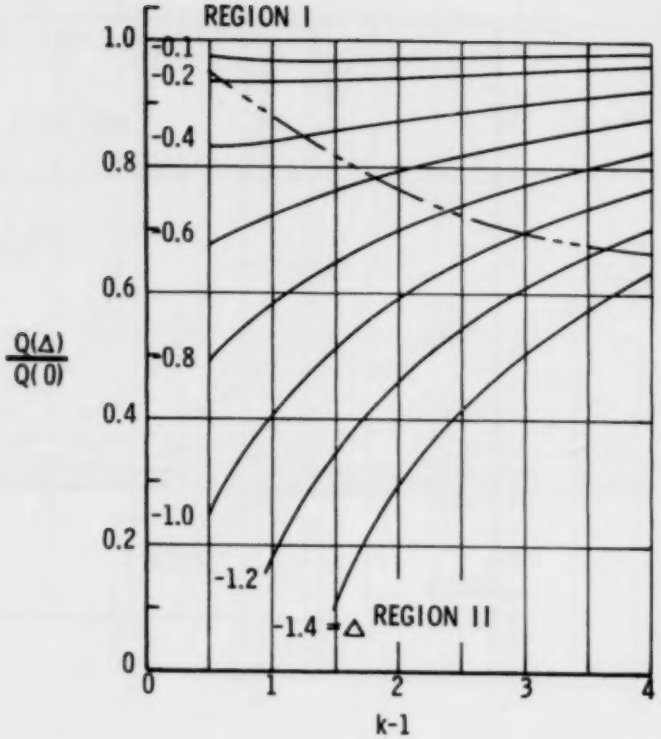


Figure 13. Flow Rate Ratio Curves for Negative Amplitude Ratios.

2

A Two-Dimensional Theory of Stress Concentration Around Voids

By Leonard G. Johnson
Research Laboratories
General Motors Corporation

INTRODUCTION

Present day emphasis on the various aspects of product reliability has led, in particular, to a serious consideration of stress raisers (holes, inclusions, etc.) in engineering materials. These stress raisers can very well be one of the chief causes of early failures in service. Therefore it behooves the materials engineer to become familiar with the magnitude of stress concentration produced by holes and inclusions of various shapes and properties. The present investigation will be confined to the problem of determining the Cartesian components of elastic stress around a hole in a plate.

STATEMENT OF THE PROBLEM

We assume a plate is subjected to a constant two-dimensional stress system whose Cartesian components are

(1) S_x = Normal stress in x-direction (horizontal) (1)

(2) S_y = Normal stress in y-direction (vertical) (2)

(3) T_{xy} = Shear stress component (3)

We represent this by the type of sketch shown in Figure 1. At great distances from the hole the components of stress are (S_x, S_y, T_{xy}). However, in the neighborhood of the hole the stress components will be considerably different from (S_x, S_y, T_{xy}). In general, let us assume the components of stress at any point (x,y) are (σ_x, σ_y, τ_{xy}). Then if the origin of our coordinate system is assumed to be at the centroid of the hole, we can state that

$$\lim_{r \to \infty} \sigma_x = S_x \tag{4}$$

$$\lim_{r \to \infty} \sigma_y = S_y \qquad (r = \sqrt{x^2 + y^2}) \tag{5}$$

$$\lim_{r \to \infty} \tau_{xy} = T_{xy} \tag{6}$$

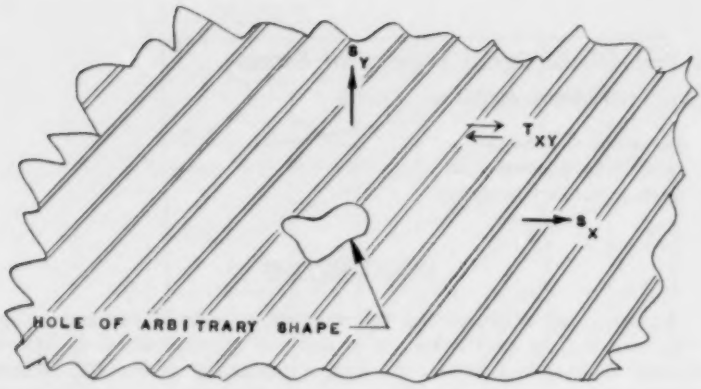


Figure 1

Let Figure 2 represent an enlarged view of the hole. Suppose the equation of the boundary of the hole is $y = B(x)$. We are assuming a no-load boundary such that for each (x,y) on $y = B(x)$

$$\text{Radial stress} = \sigma_r = 0 \tag{7}$$

$$\text{Shear stress} = \tau_{r\theta} = 0 \tag{8}$$

Converting these into Cartesian components $({}_B\sigma_x, {}_B\sigma_y, {}_B\tau_{xy})$ on the boundary of the hole we obtain the following conditions along $y = B(x)$:

$${}_B\sigma_y / {}_B\sigma_x = [B'(x)]^2 \tag{9}$$

$$\left(B'(x) = \frac{dy}{dx}\right)$$

$${}_B\sigma_x \cdot {}_B\sigma_y = {}_B\tau_{xy}^2 \tag{10}$$

STRESS FUNCTION NOTATION

It is well known [Ref. 1] that the Cartesian components of elastic stress can be represented by the second partial derivatives of a function ϕ which satisfies boundary conditions and the biharmonic partial differential equation

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \tag{11}$$

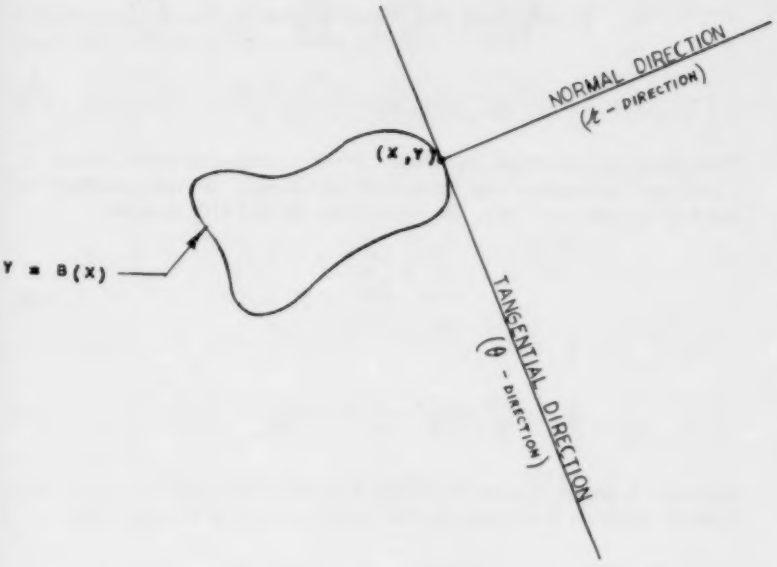


Figure 2

Then the stress components are

$$ {}_0\sigma_x = \frac{\partial^2 \phi}{\partial y^2} \tag{12} $$

$$ {}_0\sigma_y = \frac{\partial^2 \phi}{\partial x^2} \quad \text{(Assuming no body forces)} \tag{13} $$

$$ {}_0\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \, \partial y} \tag{14} $$

If we let $\phi(x,y)$ be the stress function for the plate without the hole it follows that (12), (13), and (14) must represent S_x, S_y, and T_{xy}, respectively.

Now let us assume that $\psi(x,y) = \phi(x,y) + \lambda(x,y)$ is the stress function for the plate with the hole. Then the stress components become

$$ \sigma_x = \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \lambda}{\partial y^2} = S_x + \frac{\partial^2 \lambda}{\partial y^2} \tag{15} $$

$$\sigma_y = \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \lambda}{\partial x^2} = S_y + \frac{\partial^2 \lambda}{\partial x^2} \tag{16}$$

$$\tau_{xy} = -\frac{\partial^2 \psi}{\partial x \partial y} = -\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \lambda}{\partial x \partial y} = T_{xy} - \frac{\partial^2 \lambda}{\partial x \partial y} \tag{17}$$

This form is assumed in order to retain the Cartesian frame of reference throughout the investigation. Hence, on the boundary of the hole (along $y = B(x)$), the conditions (9) and (10) become

$$\frac{S_y + \frac{\partial^2 \lambda}{\partial x^2}}{S_x + \frac{\partial^2 \lambda}{\partial y^2}} = [B'(x)]^2 \tag{18}$$

$$\left(S_x + \frac{\partial^2 \lambda}{\partial y^2}\right)\left(S_y + \frac{\partial^2 \lambda}{\partial x^2}\right) = \left(T_{xy} - \frac{\partial^2 \lambda}{\partial x \partial y}\right)^2 \tag{19}$$

Although λ is not readily discovered in any particular problem, this form of solution does indicate one possible avenue of approach.

SPECIAL CASE OF A CIRCULAR HOLE

For a circular hole of radius $a > 0$, the equation of the boundary is

$$x^2 + y^2 = a^2 \tag{20}$$

For the locus (20),

$$\frac{dy}{dx} = B'(x) = -\frac{x}{y} \tag{21}$$

Using (21) in conditions (18) and (19), we find that we can take

$$\begin{aligned} \lambda(x,y) = {} & S_x \left[\frac{a^2x^2}{x^2+y^2} - \frac{1}{4}a^4 \frac{x^2-y^2}{(x^2+y^2)^2} - \frac{1}{4}a^2 \ln(x^2+y^2)\right] \\ & + S_y \left[\frac{a^2y^2}{x^2+y^2} + \frac{1}{4}a^4 \frac{x^2-y^2}{(x^2+y^2)^2} - \frac{1}{4}a^2 \ln(x^2+y^2)\right] \\ & + T_{xy}\left[\frac{2a^2xy}{x^2+y^2} - \frac{a^4xy}{(x^2+y^2)^2}\right]. \end{aligned} \tag{22}$$

(S_x, S_y, and T_{xy} are constants.)

Substituting the partial derivatives of (22) into (15), (16), and (17) we obtain the stress components ($a \neq 0$, $x^2 + y^2 \geq a^2$)

$$\sigma_x = S_x \left[1 + \frac{3}{2} a^4 \frac{x^4 - 6x^2y^2 + y^4}{(x^2+y^2)^4} - a^2 \frac{x^4 - 6x^2y^2 + y^4}{(x^2+y^2)^3} - \frac{3}{2} a^2 \frac{x^2 - y^2}{(x^2+y^2)^2} \right]$$

$$+ S_y \left[-\frac{3}{2} a^4 \frac{x^4 - 6x^2y^2 + y^4}{(x^2+y^2)^4} + a^2 \frac{x^4 - 6x^2y^2 + y^4}{(x^2+y^2)^3} + \frac{1}{2} a^2 \frac{x^2 - y^2}{(x^2+y^2)^2} \right]$$

$$+ T_{xy} \left[\frac{12a^4 xy(x^2 - y^2)}{(x^2+y^2)^4} - \frac{8a^2 xy(x^2 - y^2)}{(x^2+y^2)^3} - \frac{4a^2 xy}{(x^2+y^2)^2} \right] \qquad (23)$$

$$\sigma_y = S_x \left[-\frac{3}{2} a^4 \frac{x^4 - 6x^2y^2 + y^4}{(x^2+y^2)^4} + a^2 \frac{x^4 - 6x^2y^2 + y^4}{(x^2+y^2)^3} - \frac{1}{2} a^2 \frac{x^2 - y^2}{(x^2+y^2)^2} \right]$$

$$+ S_y \left[1 + \frac{3}{2} a^4 \frac{x^4 - 6x^2y^2 + y^4}{(x^2+y^2)^4} - a^2 \frac{x^4 - 6x^2y^2 + y^4}{(x^2+y^2)^3} + \frac{3}{2} a^2 \frac{x^2 - y^2}{(x^2+y^2)^2} \right]$$

$$+ T_{xy} \left[\frac{-12a^4 xy(x^2 - y^2)}{(x^2+y^2)^4} + \frac{8a^2 xy(x^2 - y^2)}{(x^2+y^2)^3} - \frac{4a^2 xy}{(x^2+y^2)^2} \right] \qquad (24)$$

$$\tau_{xy} = S_x \left[6a^4 \frac{xy(x^2 - y^2)}{(x^2+y^2)^4} - 4a^2 \frac{xy(x^2 - y^2)}{(x^2+y^2)^3} - a^2 \frac{xy}{(x^2+y^2)^2} \right]$$

$$+ S_y \left[-6a^4 \frac{xy(x^2 - y^2)}{(x^2+y^2)^4} + 4a^2 \frac{xy(x^2 - y^2)}{(x^2+y^2)^3} - a^2 \frac{xy}{(x^2+y^2)^2} \right]$$

$$+ T_{xy} \left[1 - 3a^4 \frac{x^4 - 6x^2y^2 + y^4}{(x^2+y^2)^4} + 2a^2 \frac{x^4 - 6x^2y^2 + y^4}{(x^2+y^2)^3} \right] \qquad (25)$$

On the boundary of the circular hole (where $x^2 + y^2 = a^2$) these components simplify to

$$_B\sigma_x = S_x \left(\frac{y}{a}\right)^2 \left[3 - 4\left(\frac{x}{a}\right)^2 \right] + S_y \left(\frac{y}{a}\right)^2 \left[3 - 4\left(\frac{y}{a}\right)^2 \right] - 8 T_{xy} \left(\frac{x}{a}\right)\left(\frac{y}{a}\right)^3 \qquad (26)$$

$(a \neq 0)$

$$_B\sigma_y = S_x \left(\frac{x}{a}\right)^2 \left[3 - 4\left(\frac{x}{a}\right)^2 \right] + S_y \left(\frac{x}{a}\right)^2 \left[3 - 4\left(\frac{y}{a}\right)^2 \right] - 8 T_{xy} \left(\frac{x}{a}\right)^3 \left(\frac{y}{a}\right) \qquad (27)$$

$(x^2 + y^2 = a^2)$

$$_B\tau_{xy} = -S_x \left(\frac{xy}{a^2}\right) \left[3 - 4\left(\frac{x}{a}\right)^2 \right] - S_y \left(\frac{xy}{a^2}\right) \left[3 - 4\left(\frac{y}{a}\right)^2 \right] + 8 T_{xy} \left(\frac{x}{a}\right)^2 \left(\frac{y}{a}\right)^2 \qquad (28)$$

FORMULAS FOR "EFFECTIVE STRESS" ON A VOID BOUNDARY

For two-dimensional problems involving holes in very thin plates (plane stress) or very thick plates (plane strain), it is often of interest to know the value of the "effective stress" defined by

$$\sigma_{eff.}^2 = \frac{1}{2}\Big[(\sigma_x - \sigma_y)^2 + (\sigma_x - \sigma_z)^2 + (\sigma_y - \sigma_z)^2 + 6\,(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2) \Big] \tag{29}$$

Since for *plane stress*: $\sigma_z = \tau_{xz} = \tau_{yz} = 0$, we obtain

$$\sigma_{eff.}^2 = (\sigma_x + \sigma_y)^2 + 3\,(\tau_{xy}^2 - \sigma_x \sigma_y) \tag{30}$$

For *plane strain*, where $\sigma_z = \nu(\sigma_x + \sigma_y)$ and $\tau_{xz} = \tau_{yz} = 0$, (29) reduces to ($\nu$ = Poisson's ratio)

$$\sigma_{eff.}^2 = (1 - \nu + \nu^2)(\sigma_x + \sigma_y)^2 + 3\,(\tau_{xy}^2 - \sigma_x \sigma_y) \tag{31}$$

On the void boundary ${}_B\tau_{xy}^2 - {}_B\sigma_x\,{}_B\sigma_y = 0$. (See (10).)

Hence,

For plane stress: $${}_B\sigma_{eff.} = \left| {}_B\sigma_x + {}_B\sigma_y \right| . \tag{32}$$

For plane strain: $${}_B\sigma_{eff.} = \left| {}_B\sigma_x + {}_B\sigma_y \right| \sqrt{1 - \nu + \nu^2} . \tag{33}$$

CONCLUSION

For any void, $y = B(x)$, in a plate subjected to stresses S_x, S_y, T_{xy} it is possible to select an auxiliary stress function $\lambda(x,y)$ which will satisfy (18) and (19). The second partial derivatives of $\lambda(x,y)$ when added to S_x, S_y, T_{xy} in accordance with (15), (16), and (17) will then give the Cartesian stress components anywhere around the void, and questions of stress concentration can be answered.

REFERENCES

[1] Timoshenko, S. and Goodier, J. N.: Theory of Elasticity, second edition, McGraw-Hill, 1951. (Page 16.)

3

Stability Study of a Fluid Flow Problem with Flow Reversal

By Richard L. Justice
Research Laboratories
General Motors Corporation

In considering the problem of fluid flow an interesting mathematical problem arises when the flow reverses. In order to gain a thorough understanding of this process a very simple two volume model will be studied. This model is designed to introduce into the mathematics only those problems of instability and not consider the problem of heat transfer. Insight gained from this problem may well lead to applications in real problems involving mathematical instability.

TABLE OF SYMBOLS

A	- Cross Sectional Area
C_V	- Specific heat at constant volume
C_P	- Specific heat at constant pressure
D	- Diameter of tube
F	- Friction factor of tube
g	- Gravity constant
k	- Ratio of specific heats
$K, K^1, \tilde{K}$	- System constants
M	- Mass
N_R	- Reynolds' number
P	- Pressure
Q	- Heat
R	- Gas constant
T	- Temperature
U	- Gas velocity
V	- Volume
W	- Mass rate of flow
W_K	- Work
θ	- Time
ρ	- Density
μ	- Viscosity

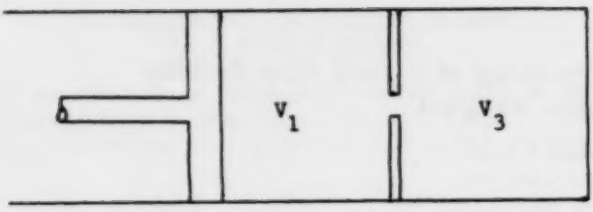


Figure 1

Applying the first law of thermodynamics to the volumes as shown in Figure 1, the following equations are obtained:

$$\frac{d\left(\frac{C_V}{R} P_1 V_1\right)}{d\theta} = C_P T_{in} \frac{d M_{in}}{d\theta} - C_P T_{out} \frac{d M_{out}}{d\theta} + \frac{d Q}{d\theta} - \frac{d W_{K1}}{d\theta} \quad (1)$$

$$\frac{d\left(\frac{C_V}{R} P_3 V_3\right)}{d\theta} = C_P T_{in} \frac{d M_{in}}{d\theta} - C_P T_{out} \frac{d M_{out}}{d\theta} + \frac{d Q}{d\theta} - \frac{d W_{K3}}{d\theta} \quad (2)$$

Since the process is considered adiabatic, $\frac{dQ}{d\theta} = 0$. Further, let $\frac{dM}{d\theta} = W$ (flow rate of the medium), and $\frac{dW_{K1}}{d\theta} = P_1 \frac{dV_1}{d\theta}$ and $\frac{dW_{K3}}{d\theta} = 0$ (since V_3 is constant). At a given instant if the piston is moving to the right, equations (1) and (2) become

$$\frac{C_V}{R} \frac{d (P_1 V_1)}{d\theta} = - C_P T_{out} W_{out} - P_1 \frac{d V_1}{d\theta}$$

$$\frac{C_V}{R} \frac{d (P_3 V_3)}{d\theta} = C_P T_{in} W_{in}$$

Performing the differentiation as indicated yields

$$\frac{d P_1}{d\theta} = \frac{1}{V_1}\left[\frac{R}{C_V}\left(- C_P T_{out} W_{out} - P_1 \frac{d V_1}{d\theta}\right) - P_1 \frac{d V_1}{d\theta}\right]$$

$$\frac{d P_3}{d\theta} = \frac{1}{V_3}\left(\frac{R C_P}{C_V} T_{in} W_{in}\right)$$

or

$$\frac{d P_1}{d\theta} = \frac{1}{V_1}\left(- k R T_{out} W_{out} - k P_1 \frac{d V_1}{d\theta}\right) \tag{3}$$

$$\frac{d P_3}{d\theta} = \frac{1}{V_3} k R T_{in} W_{in} \tag{4}$$

Observe that as the piston traverses its allowed path, several conditions with regard to the pressures may exist; namely, $P_1 > P_3$ or $P_3 > P_1$ with either turbulent or laminar flow depending upon the magnitude of the inequality. For the moment let us assume that $P_1 > P_3$ and the inequality is sufficient to develop turbulent flow.

Define the following dimensionless parameter.

$$P_2 = \sqrt{1 - P_3/P_1}$$

From this, then

$$\frac{d P_2}{d\theta} = \frac{1}{2 P_2}\left(- \frac{1}{P_1}\frac{d P_3}{d\theta} + \frac{P_3}{P_1^2}\frac{d P_1}{d\theta}\right) \tag{5}$$

and

$$P_3 = P_1 (1 - P_2^2)$$

Next observe that for turbulent flow

$$P_1 - P_3 = \frac{f L U^2 \rho}{2 g D}, \quad U = \frac{W}{\rho A}$$

$$W = \frac{K}{\sqrt{T_1}}\sqrt{P_1 (P_1 - P_3)} \quad K = \sqrt{\frac{2g}{\frac{f L}{A^2 D} R}}$$

$$W = \frac{K P_1 P_2}{\sqrt{T_1}}$$

By substituting in (5) the following expression is obtained

$$\frac{d P_2}{d\theta} = \frac{1}{2 P_2}\left[- \frac{1}{P_1}\left(\frac{1}{V_3} R k T_1 \frac{K P_1 P_2}{\sqrt{T_1}}\right) + \frac{P_1 (1 - P_2^2)}{P_1^2}\left(\frac{- k R K T_1 P_1 P_2}{V_1 \sqrt{T_1}} - \frac{k P_1}{V_1}\frac{d V_1}{d\theta}\right)\right]$$

$$\frac{d P_2}{d\theta} = - \frac{k R \sqrt{T_1} K}{2 V_1} \left(\frac{V_1}{V_3} + 1 - P_2^2 \right) - \frac{k (1 - P_2^2)}{2 P_2 V_1} \frac{d V_1}{d\theta}$$

Since $M_1 + M_3 = M_T$ and $\frac{d M_1}{d\theta} = - W_{in} = W_{out}$ we may now summarize the three essential equations for $P_1 > P_3$ turbulent flow.

$$\frac{d P_1}{d\theta} = - \frac{k R K \sqrt{T_1} P_1 P_2}{V_1} - \frac{k P_1}{V_1} \frac{d V_1}{d\theta} \tag{6}$$

$$\frac{d P_2}{d\theta} = - \frac{k R K \sqrt{T_1}}{2 V_1} \left(\frac{V_1}{V_3} + 1 - P_2^2 \right) - \frac{k (1 - P_2^2)}{2 P_2 V_1} \frac{d V_1}{d\theta} \tag{7}$$

$$\frac{d M_1}{d\theta} = - \frac{K P_1 P_2}{\sqrt{T_1}} \tag{8}$$

Next consider the case of turbulent flow for $P_3 > P_1$. Define

$$P_2 = \sqrt{1 - P_1 / P_3}$$

and hence,

$$P_3 = \frac{P_1}{1 - P_2^2}$$

The following development parallels the above.

$$\frac{d P_1}{d\theta} = \frac{1}{V_1} \left(k R T_3 W_{in} - k P_1 \frac{d V_1}{d\theta} \right)$$

$$\frac{d P_3}{d\theta} = \frac{1}{V_3} k R T_3 W_{out}$$

$$W_{in} = \frac{\overline{K}}{\sqrt{T_3}} \sqrt{P_3 (P_3 - P_1)}$$

$$W_{in} = \frac{\overline{K}}{\sqrt{T_3}} P_3 P_2$$

$$W_{in} = \frac{\overline{K} P_1 P_2}{\sqrt{T_3} (1 - P_2^2)}$$

$$\frac{d P_2}{d\theta} = \frac{1}{2 P_2} \left(- \frac{1}{P_3} \frac{d P_1}{d\theta} + \frac{P_1}{P_3^2} \frac{d P_3}{d\theta} \right)$$

$$\frac{d P_2}{d\theta} = \frac{1}{2 P_2}\left[-\frac{1}{P_3}\left(\frac{1}{V_1} k R \bar{K} \sqrt{T_3} P_3 P_2 - \frac{k P_1}{V_1}\frac{d V_1}{d\theta}\right)\right.$$

$$\left. - \frac{P_1}{P_3^2}\left(\frac{k R \sqrt{T_3} P_1 P_2 \bar{K}}{V_3 (1 - P_2^2)}\right)\right]$$

$$\frac{d P_2}{d\theta} = \frac{\bar{K} k R \sqrt{T_3}}{2 V_3}\left(\frac{V_3}{V_1} + 1 - P_2\right) + \frac{k (1 - P_2^2)}{2 V_1 P_2}\frac{d V_1}{d\theta}$$

The turbulent $P_3 > P_1$ equations are summarized here

$$\frac{d P_1}{d\theta} = \frac{k R \bar{K} \sqrt{T_3} P_1 P_2}{V_1 (1 - P_2^2)} - \frac{k P_1}{V_1}\frac{d V_1}{d\theta} \tag{9}$$

$$\frac{d P_2}{d\theta} = -\frac{k R \bar{K} \sqrt{T_3}}{2 V_3}\left(\frac{V_3}{V_1} + 1 - P_2^2\right) + \frac{k (1 - P_2^2)}{2 V_1 P_2}\frac{d V_1}{d\theta} \tag{10}$$

$$\frac{d M_1}{d\theta} = \frac{\bar{K} P_1 P_2}{\sqrt{T_3} (1 - P_2^2)} \tag{11}$$

These equations (6)-(8), and (9)-(11), will not be applicable in the laminar flow region, since they are inherently unstable at the flow reversal point. This is caused by the occurrence of P_2 in the denominator of the second term in $\frac{d P_2}{d\theta}$. (Note that the $\frac{d P_1}{d\theta}$ equation will not be unstable.) Hence a new definition of the dimensionless P_2 is required.

Consider laminar flow with $P_1 > P_3$. Define

$$P_2 = 1 - \frac{P_3}{P_1}$$

Then

$$\frac{d P_2}{d\theta} = -\frac{1}{P_1}\frac{d P_3}{d\theta} + \frac{P_3}{P_1^2}\frac{d P_1}{d\theta} \tag{12}$$

where $\frac{d P_1}{d\theta}$ and $\frac{d P_3}{d\theta}$ are defined as in (3) and (4). According to such references as McAdams[1] the following relationships hold,

[1] McAdams, W. H., "Heat Transmission" 3rd Ed., McGraw-Hill, New York, 1954.

$$f = \frac{64}{N_R} = \frac{64}{\frac{D\,U\,\rho}{\mu}} = \frac{64}{\frac{D\,W}{\mu\,A}}$$

Then flow rate is obtained as follows

$$P_1 - P_3 = \frac{f\,L\,U^2\rho}{2\,g\,D}$$

$$P_1 - P_3 = \frac{64\,L\,W\,\mu}{2\,g\,D^2\,A\,\rho}$$

$$W_{out} = \frac{(P_1 - P_3)\,P_1}{P_1}\,\frac{P_1\,2g\,A\,D^2}{R\,T_1\,64\,L\,\mu}$$

$$W_{out} = \frac{P_1^2\,P_2\,K^1}{T_1\,\mu}$$

where $K^1 = \frac{2\,g\,A\,D^2}{64\,L\,R}$

Making these substitutions in (12) we have

$$\frac{d\,P_2}{d\theta} = \frac{-1}{P_1}\left(\frac{1}{V_3}\,\frac{k\,R\,K^1\,P_1^2\,P_2}{\mu}\right) + \frac{P_1\,(1 - P_2)}{P_1^2\,V_1}\left(\frac{k\,R\,K^1\,P_1^2\,P_2}{\mu} - k\,P_1\,\frac{d\,V_1}{d\theta}\right)$$

The following three equations hold for $P_1 > P_3$ laminar flow.

$$\frac{d\,P_1}{d\theta} = -\frac{P_1^2\,P_2\,R\,k\,K^1}{V_1\,\mu} - \frac{k\,P_1}{V_1}\,\frac{d\,V_1}{d\theta} \tag{13}$$

$$\frac{d\,P_2}{d\theta} = -\frac{P_1\,P_2\,R\,k\,K^1}{V_1\,\mu}\left(\frac{V_1}{V_3} + 1 - P_2\right) - \frac{k\,(1 - P_2)}{V_1}\,\frac{d\,V_1}{d\theta} \tag{14}$$

$$\frac{d\,M_1}{d\theta} = -\frac{K^1\,P_1^2\,P_2}{T_1\,\mu} \tag{15}$$

The final set of equations needed then are the laminar equations for $P_3 > P_1$. In this case the following relationships apply.

$$P_2 = 1 - \frac{P_3}{P_1} \qquad \text{(hence } P_2 < 0\text{)}$$

$$\frac{d\,P_2}{d\theta} = -\frac{1}{P_1}\,\frac{d\,P_3}{d\theta} + \frac{P_3}{P_1^2}\,\frac{d\,P_1}{d\theta}$$

$$W_{in} = \frac{\bar{K}\, P_1^2\, P_2\, (1 - P_2)}{T_3\, \mu}$$

$$\frac{d P_1}{d\theta} = - \frac{k\, R\, \bar{K}\, P_1^2 P_2\, (1 - P_2)}{V_1\, \mu} - \frac{k\, P_1}{V_1} \frac{d V_1}{d\theta} \tag{16}$$

$$\frac{d P_2}{d\theta} = - \frac{k\, R\, \bar{K}\, P_1\, P_2}{V_1\, \mu} \left(\frac{V_1}{V_3} + 1 - P_2 \right) - \frac{k\, (1 - P_2)}{V_1} \frac{d V_1}{d\theta} \tag{17}$$

$$\frac{d M_R}{d\theta} = - \frac{P_1^2 P_2\, (1 - P_2)\, \bar{K}}{T_3\, \mu} \tag{18}$$

Figure 2 illustrates the application of these four sets of equations and the points at which a decision must be made to switch sets.

In order to arrive at a satisfactory numerical solution of these equations, it is necessary to maintain two different types of accuracy. Firstly, it is necessary to hold the error within reasonable engineering accuracy, such that results could be corroborated by tests. Secondly, it is necessary to maintain an accuracy necessary for stability of the numerical technique employed to solve the differential equations. This latter one is by far the more critical one and is such that the "roots" must be located quite accurately; roots are the points of the cycle where a transition from turbulent to laminar or laminar to turbulent flow occur. The principal observation of this study is that if these roots are located with sufficient accuracy then it is not particularly difficult to maintain a decent engineering accuracy over the rest of the cycle.

With regard to the numerical techniques and difficulties, the Runge-Kutta-Gill method was used with root iterations to find the above mentioned roots. (This problem was solved with the use of an

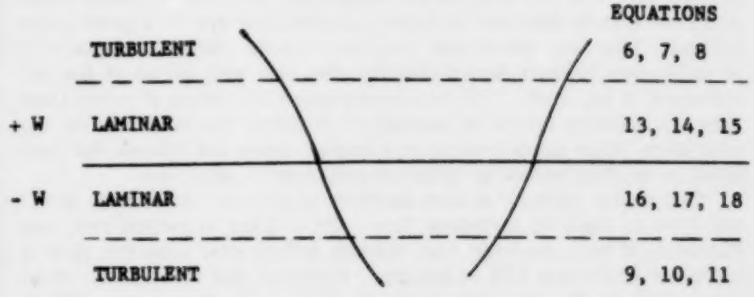


Figure 2

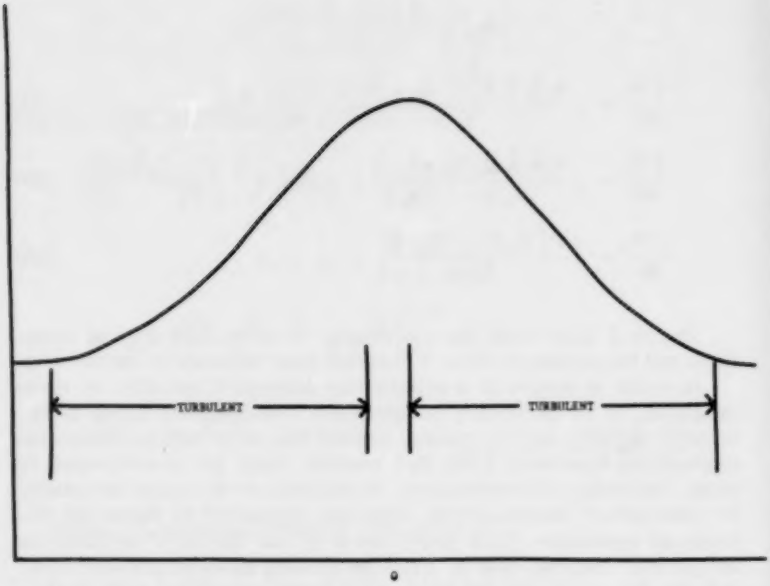


Figure 3

IBM 704). It should be noted that the location of the roots is not a function of the independent variable alone, but is also a function of the dependent variables. This can lead to the following problem. When RKG has produced a number which appears to have spanned a root, an effort is made to iterate and locate the root more accurately. By this more accurate technique the root is determined more accurately also and actually appears to move to a point later in time. This then causes the program to stop. Hence it appears to be necessary to start decreasing the step size well ahead of the occurrence of any root. This is accomplished by testing at every time step the relative merit of halving or doubling the step size on the next step. This costs heavily in compute time, but allows the machine to operate at nearly optimum step size at all times.

One of the oddities of this problem is the ratio of time in laminar flow to time in turbulent flow. On a given standard run, see Figure 3, it was observed that through 86% of real time the flow is turbulent while only 14% is laminar. However, due to stability problems, 57% of the computer time is spent in turbulent and 43% in laminar. The anamoly here is that in general the usefulness of such

a program as this is greatest in the region of greatest change in pressure and hence highest flow. Therefore, the contribution to useful results obtained from the laminar region is negligible even though it costs about half the computer time.

This problem was formulated with the hope of gaining a better understanding of the mechanism of flow reversal. The primary prerequisites of the program were to get some measure of the stability requirements and the length of time necessary to compute converged results. The problem serves primarily as an initial step towards handling n linked volumes in series with n - 2 constant volume tanks and two varying volumes. The final program of interest then will be to yield a complete pressure, temperature and mass distribution picture of such a system as a function of time.

4

Computers—Large or Small?

By Robert K. Louden
System Engineer
International Business Machines Corporation
Detroit, Michigan

INTRODUCTION

If you were running a department within a company and your department required electronic data processing, would you attempt to share a large computer with other departments, or would you prefer to go it alone, with a smaller computer set up specifically for your own department? This paper is a qualitative discussion of some of the factors which affect the choice of "Computers – Large or Small?" based upon the author's experiences with various companies, some of which chose large computers and some of which chose small computers. The opinions below are the author's and do not necessarily reflect the judgments of any particular computer user.

Small computers are here defined as digital systems renting for less than $5,000.00 per month and large computers are defined as digital systems renting for more than $25,000.00 per month. It is assumed that the data processing work of the department in question can be handled on either system; if the small computer lacks either enough processing speed or memory capacity to do the work, then the decision must automatically go to the large machine. It is also assumed that both systems are typical of new equipment being installed in 1961; i.e., that the large computer is a transistorized magnetic core system with multiple tape channels and off-line input and output, and that the small computer is a transistorized magnetic core system with buffered input and output.

Cost and Throughput

In the area of direct operating cost, or answers per dollar of computer rental, the large computer generally comes out well ahead of its small competitor. This is due to the fact that both large and small computers must possess certain basic input/output and logical capabilities, and a comparatively large proportion of the small computer cost must be set aside to meet these basic requirements. The additional cost of the large computer can then be used to provide increased speed and memory capacity which will produce significantly higher answer per dollar ratios.

However, if the large computer will be located some distance away from the departments which will use it, the cost of transportation of men and information to and from the computer, plus the cost of time delays caused by the transportation problems, may be sufficient to overcome the large computer's direct cost advantage. The cost of transportation and time delays has been explored at some length by Donald C. Gause of the General Motors Truck & Coach Division of General Motors Corporation, who had developed comprehensive data on the breakeven cost of large versus small computers as a function of the large computer's distance and transportation time, considering transportation methods ranging from men on foot to telephone lines.

TELE-PROCESSING equipment which allows high speed long distance transmission of data provides a satisfactory means for the transfer of low volume high priority data, but at present most high volume Tele-Processing Systems cost about as much as a typical small computer system.

The greater speed and capacity of the large computer will enable it to handle big jobs which might develop in the future, while the small computer may not be able to do so, due to its more limited memory size. Therefore, the likelihood of such big jobs developing in the department should be ascertained, if possible, before a decision on computing equipment is made.

Communications

Computer communication refers to the task of instructing the computer how to solve problems (called programming) and then checking and testing the instructions until they produce correct answers (called debugging). A programming language or code is used to write the instructions in a form which the computer can translate into basic computer operations such as add, subtract, etc.

Usually, the large computers will utilize languages which are very general and very powerful. The small computers will use languages which are (by comparison) simple and ridigly defined. This usually means that experienced programmers working on large complex problems will have an easier time of it with the big machine languages, but that novice programmers on small problems make fewer mistakes (and therefore learn faster) with the simpler languages of the small machines, provided that the problems to be solved are not too large for the small computers to handle conveniently. If, however, the problems strain the memory size of the small computer to the point where the programmer must spend an appreciable part of his time figuring out a memory allocation scheme to fit the problems into the small computer, then the advantage of language simplicity will be of little value.

The large computer will generally have superior diagnostics (tools for analyzing program errors) and more extensive subroutine facilities than the small computer. These aids are very much needed on the large, complex problems which big computers usually solve; large diagnostic and subroutine packages are not so essential on the smaller, more straight forward problems for which small computers are designed.

Access and Turn Around Time

In the area of access and turn around time, the small computer appears to come out ahead, since it will usually be located much closer to the department than would be a large computer which would be shared between departments, and the access of various problems to the small computer would be entirely controlled by the department, whereas on the large computer this access would be controlled by a priority system designed to accommodate the varying needs of several different departments. In general, the farther removed the large computer is from the department which must use it, the greater will be the advantage in access and turn around time of the small machine. However, it should be borne in mind that this will only hold true for problems which are small enough to be conveniently handled upon the small computer.

Particularly during checkout of a new program, it is desirable for the programmer to be stationed near the computer so that he will be in a position to diagnose the results of his program and send it back to the computer as soon as possible. This usually results in sending the programmer to the computer for a sizeable period of time if the computer is remotely located from the programmer's department, and adds to the cost of operating a remotely located computer installation.

The complete control of access by one department means that the department, of course, will have unlimited access to the machine, but that various individuals and groups within the department will still face a queuing or priority problem similar to that which the department would face in using a larger machine. The higher the load factor or the greater the percentage of time which the computer is actually in use out of a 24 hour day, the longer will be the turn around time and the more time will be lost by individuals waiting to make use of the computer. This will tend to be true on small systems as well as large systems, so that the access of any computer can only really be optimized if free time on the computer is available to take care of unexpected surges in demand.

Administration

The administration problem as to whether to try to combine operations on a single large computer between various departments or to install separate smaller computers for the various departments is usually the most important single factor in determining the feasibility of separate or joint operations. If departments fulfilling wholly different functions within a company are to share a single data processing installation, there must usually be some sort of a referee or administrative group set up to allocate computer time and decide the priority of various computer projects. If such a group does not exist, then the various departments will have to cooperate completely among themselves and if the departments tend to follow lines of authority on opposite sides of the company's organization chart (as for example in the case of Product Engineering and Accounting) then they may indeed have no well defined body to turn to to settle such priority and scheduling disputes as may be expected to occur. For instance, the problem of what to do when a chief engineer insists on receiving test data by 3:00 p.m. and at the same time the hourly payroll is running three hours late has seldom been successfully resolved, except by the intervention of someone in a position to arbitrate between the two departments involved. This sort of arbitration and compromise, of course, is not necessary if a small computer is used to serve each department.

Since the big computer will require a certain amount of cooperation and indeed standardization between departments and since it has the capacity to handle very large problems, it also offers the potentiality of large scale data processing applications which will cut across departmental lines and offer efficiencies and economies which would not be possible in an independent solution of the problems of any one department. This, of course, is one of the brightest goals of large scale data processing, to integrate the overall data processing requirements of entire companies.

The price of such across-the-board integration of data processing is frequently a rather drastic reorganization of lines of departmental responsibility within the company and many companies have not yet found it practical to take a step of this type in an effort to integrate their data processing. When examined in this light, the choice of several small computers appears as a fairly conservative way to achieve reasonably limited goals, whereas the decision to pool the departments on a single large computer appears as a more radical way to achieve larger returns from the data processing investment.

The fundamental management policies of the company involved must necessarily play a large part in this decision. For instance,

if the company believes in a policy of complete decentralization and individual responsibility for its departments, it may tend to shy away from integrating the data processing of these departments into a single installation. Conversely, if the company is trying to centralize management operations, it will probably find that integrated data processing on a single large machine will be a very powerful step in the direction of such centralized operations.

In most small versus large computer decisions, administrative considerations outweigh the technical considerations of direct cost, throughput and access time and tend to establish a pattern of centralized or decentralized data processing which reflects the management philosophy of the company involved.

5

Par
on

By
Re
G

curv
lowe
func

A di
$\rho''(6$

rest
of c
com
are
term
prof
rect
cam
ord

is h
ous
reg
tan
the
rota

is c
that

5

Parameters Needed for Stress Calculations on Cams

By Robert Davies
Research Laboratories Division
General Motors Corporation

Abstract. In this paper formulas are derived for the radius of curvature of a cam and for the distance along the cam and its follower to the point of contact. If the lift ρ of a flat follower is the function $\rho(\theta)$ of cam rotation θ, then

$$\text{radius of curvature} = \rho''(\theta) + \rho(\theta),$$

$$\text{distance along follower} = \rho'(\theta),$$

$$\text{distance along cam} = \rho'(\theta) + \int_0^\theta \rho\,(\theta)\,d\theta\ .$$

A discussion is given of numerical means of calculating $\rho'(\theta)$ and $\rho''(\theta)$ when the function $\rho(\theta)$ is defined by a table.

Introduction. The stresses in a cam follower produced by its resting against the cam depend on the force of contact, on the radius of curvature of the cam, and on its width. The width and the static component of the force can be measured directly, but large errors are inherent in any scheme in which the radius of curvature is determined by constructing osculating circles on a drawing of the cam profile. It is more accurate to calculate the radius of curvature directly from the cam-lift data or from the data used to grind the cam. Also, the acceleration of the cam follower must be known in order to calculate the dynamic force between the follower and cam.

In examining the pitting of fatigued cams and cam followers, it is helpful to know what points on the surfaces are in contact at various cam rotations in order to determine the stress to which a pitted region is subjected. It is desirable to be able to calculate the distance along the surface of the cam to the point of contact, as well as the distance across the face of the follower, as a function of cam rotation.

Generally cams are prescribed by giving the amount of lift that is obtained as a function of cam rotation or by giving the amount that a cutter or grinding wheel is to be advanced toward the cam

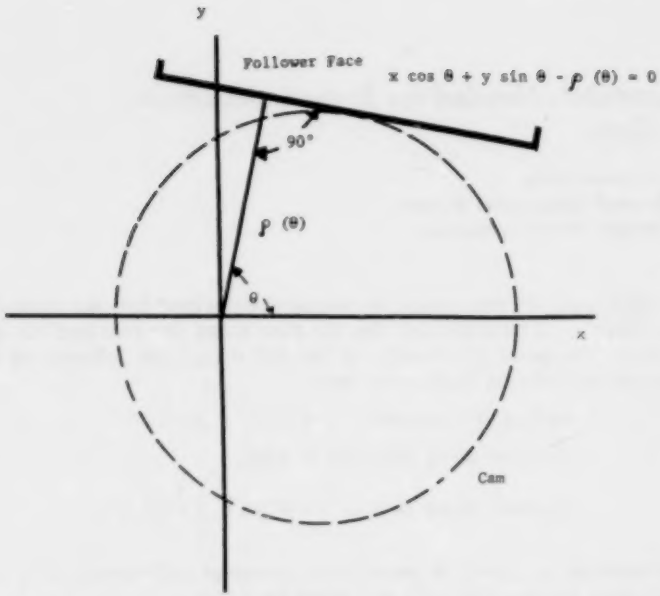


Figure 1. The Equation for the Follower Face in Normal Form.

axis as a function of cam rotation. The prescription is not equivalent to giving the rectangular or polar co-ordinates of the cam profile.

Equations for the Cam Profile. The face of a flat follower or grinding wheel can be represented by the "normal" form for the equation of a straight line, that is, by

$$x \cos \theta + y \sin \theta - \rho(\theta) = 0,$$

in which the x,y-axes are perpendicular to the axis of rotation and rotate with the cam and in which the origin is on the axis of rotation. The perpendicular distance from the straight line to the origin is ρ, and θ is the angle between the perpendicular and the x-axis.

If the cam follower is circular in cross section (a roller), or if the grinding wheel is circular, then the equation is

$$[x - \rho(\theta) \cos \theta]^2 + [y - \rho(\theta) \sin \theta]^2 = r^2,$$

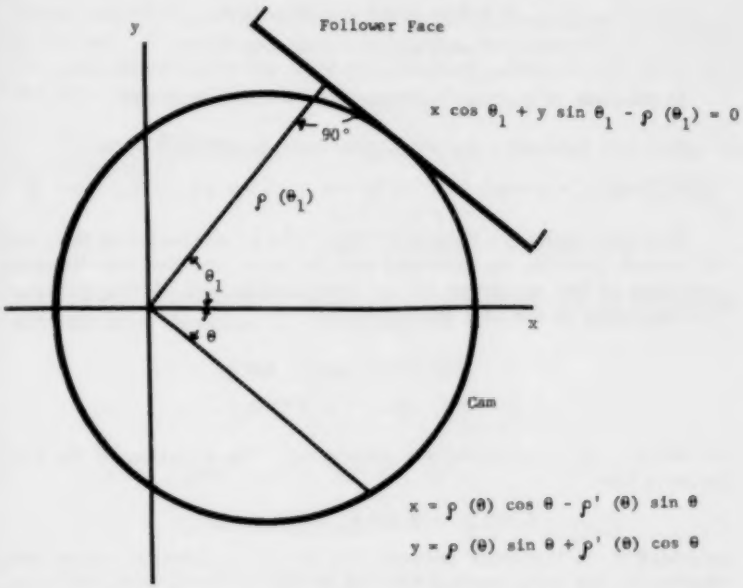


Figure 2. The Coordinates of a Point on the Face of the Cam and the Equation of the Follower.

in which the same axes are used but in which ρ is the distance from the cam axis to the center of the roller, θ is the angle that a line between these points makes with the x-axis, and r is the radius of the roller.

The cam profile is the envelope of all of these straight lines or circles. For a flat follower, the equation of the envelope is

$$\begin{aligned} x \cos\theta + y \sin\theta - \rho(\theta) &= 0 . \\ -x \sin\theta + y \cos\theta - \rho'(\theta) &= 0 . \end{aligned} \tag{1}$$

The first equation represents the cam follower. The second is obtained from it by differentiating with respect to the parameter θ, the derivatives being indicated by primes.

By eliminating θ, the two equations can be reduced to give a single equation in x and y only (ρ is given as a function of θ and is therefore eliminated when θ is eliminated). The resulting equation is the equation of the cam profile. However, it is more convenient to work with Equation (1) in the parametric form:

$$x = \rho(\theta) \cos \theta - \rho'(\theta) \sin \theta ,$$
$$y = \rho(\theta) \sin \theta + \rho'(\theta) \cos \theta . \tag{2}$$

In the case of a roller follower, the two equations are

$$[x - \rho(\theta) \cos \theta]^2 + [y - \rho(\theta) \sin \theta]^2 = r^2 ,$$
$$\rho(\theta) [x \sin \theta - y \cos \theta] - \rho'(\theta) [x \cos \theta + y \sin \theta] + \rho(\theta)\rho'(\theta) = 0 .$$

Distance Along the Follower Face. The co-ordinates of the point of contact between the follower and the cam are the simultaneous solutions of the equations of the cam profile and of the follower. The equations of the cam profile are

$$x = \rho(\theta) \cos \theta - \rho'(\theta) \sin \theta ,$$
$$y = \rho(\theta) \sin \theta + \rho'(\theta) \cos \theta .$$

in which θ is an unrestrained parameter. The equation of the cam follower is

$$x \cos \theta_1 + y \sin \theta_1 - \rho(\theta_1) = 0 ,$$

in which θ_1 is the angle between the x-axis, a direction fixed with respect to the cam, and the normal to the cam follower face, a direction fixed in space. In other words, θ_1 is the angle of cam rotation. The simultaneous solutions of these equations are

$$x = \rho(\theta_1) \cos \theta_1 - \rho'(\theta_1) \sin \theta_1 ,$$
$$y = \rho(\theta_1) \sin \theta_1 + \rho'(\theta_1) \cos \theta_1 . \tag{3}$$

It is evident that the point (x,y) on the cam profile, obtained from Equation 2 for a given value of the parameter θ, is the point in contact with the follower for the cam rotation θ.

The perpendicular from the axis of the cam to the follower face makes an angle θ_1 with the x-axis, and its length is $\rho(\theta_1)$. The co-ordinates of the intersection with the follower face are then

$$x = \rho(\theta_1) \cos \theta_1 ,$$
$$y = \rho(\theta_1) \sin \theta_1 . \tag{4}$$

This intersection is a convenient point from which to measure distances along the follower face. The distance between the points given by Equations (3) and (4) is

$$D = \rho'(\theta_1) ,$$

which, therefore, is the distance along the follower face to the point of contact. At maximum lift, $\rho'(\theta)$ is zero; consequently, distance is being measured from the point of contact at maximum lift. It is also the point closest to the cam axis.

Radius of Curvature. In rectangular co-ordinates the radius of curvature R is

$$R = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'}$$

for a curve in parametric form. The derivatives can be calculated directly from Equation (2). They are

$$x' = -[\rho(\theta) + \rho''(\theta)] \sin\theta,$$

$$y' = +[\rho(\theta) + \rho''(\theta)] \cos\theta;$$

$$x'' = -[\rho(\theta) + \rho''(\theta)] \cos\theta - [\rho'(\theta) + \rho'''(\theta)] \sin\theta,$$

$$y'' = -[\rho(\theta) + \rho''(\theta)] \sin\theta + [\rho'(\theta) + \rho'''(\theta)] \cos\theta.$$

Substitution in the equation for R gives

$$R = \rho(\theta) + \rho''(\theta),$$

which is the radius of curvature of the cam profile at (x,y), the point of contact for cam rotation θ.

Distance Along Cam Surface. The arc length along a curve that is given in parametric form is

$$L = \int_{\theta_0}^{\theta} \sqrt{x'^2 + y'^2}\, d\theta,$$

in which θ_0 is the value of the parameter at the point from which distance is measured. For the cam profile represented by Equation (2),

$$L = \rho'(\theta) - \rho'(\theta_0) + \int_{\theta_0}^{\theta} \rho(\theta)\, d\theta$$

is the distance between the points of contact for the positions θ and θ_0 of the cam. Often, however, the distance along the cam is measured from the point of contact at maximum lift, and the rotation of

the cam is measured from the angle that gives maximum lift, so that $\theta_0 = 0$ and $\rho'(0) = 0$. In this case,

$$L = \rho'(\theta) + \int_0^\theta \rho(\theta)\, d\theta$$

is the distance along the cam surface to the point of contact.

Acceleration of the Cam Follower. The displacement ρ of the follower is

$$\rho = \rho(\theta);$$

the velocity and acceleration are therefore

$$\dot{\rho} = \rho'(\theta)\dot{\theta},$$
$$\ddot{\rho} = \rho'(\theta)\ddot{\theta} + \rho''(\theta)\dot{\theta}^2,$$

in which dots denote derivatives with respect to time.

For a cam that turns at uniform velocity, $\ddot{\theta}$ is zero and the acceleration of the cam follower is

$$\ddot{\rho} = \rho''(\theta)\dot{\theta}^2.$$

Calculation of $\rho'(\theta)$ and $\rho''(\theta)$. The derivative of a function is defined in such a way that it can be calculated nowhere except at limit points of the set over which the function is defined. Functions that are defined for every point in an interval *may* have a derivative in the interval; but functions defined over isolated sets of points – and these include all functions defined by tables – cannot have derivatives. Consequently, the function $\rho(\theta)$, if it is defined by a table, must be extended to the intervals between the tabular entries before the derivatives $\rho'(\theta)$ and $\rho''(\theta)$ can be calculated.

The cam consists of a sequence of minute flat facets when it is machined with a flat cutter. For such a cam it is possible to calculate the lift for *all* angles between the values in the table and then to calculate the corresponding derivatives. However, this method is not a very realistic one for interpolation because, in use, these facets soon loose their sharp edges and wear into a smoothly curved surface.

Polynomials are frequently used to interpolate between tabular values when a smooth function is desired, since they are easy to evaluate and to differentiate. The interpolation polynomial for $\rho(\theta)$ can be found in the following way if the values of $\rho(\theta)$ are given for equally spaced values of θ. Table 1 is a difference table; each entry

Table 1 - Example of Difference Table

n	ρ	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
-3	+23148						
		+3801					
-2	+26949		+396				
		+4197		-14			
-1	+31146		+382		-7		
		+4579		-21		+1	
0	+35725		+361		-6		0
		+4940		-27		+1	
+1	+40665		+334		-5		
		+5274		-32			
+2	+45939		+302				
		+5576					
+3	+51515						

is obtained from the column to the left of it by subtracting the entry immediately above it from the entry immediately below it. Table 2 illustrates the notation that is generally used. For the differences in Table 2,

$$\Delta^i \rho_m = \Delta^{i-1} \rho_{m+\frac{1}{2}} - \Delta^{i-1} \rho_{m-\frac{1}{2}}$$

The two left columns represent the value of n and ρ. The variable n is used in place of θ in order to simplify some later equations. It is chosen to have the value zero in the middle of the table and to increase and decrease in increments of one. They are related by the linear transformation

$$(5) \qquad n = \frac{\theta - \theta_0}{\Delta\theta}, \qquad \theta = n\Delta\theta + \theta_0,$$

in which $\Delta\theta$ is the increment in θ between entries in the table.

There are no odd-order differences with subscript zero, but they can be introduced as the averages

$$\Delta^i \rho_0 = \frac{1}{2}\Delta^i \rho_{-\frac{1}{2}} + \frac{1}{2}\Delta^i \rho_{+\frac{1}{2}}$$

Table 2 - General Difference Table

n	ρ	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
-3	ρ_{-3}						
		$\Delta\rho_{-2\frac{1}{2}}$					
-2	ρ_{-2}		$\Delta^2\rho_{-2}$				
		$\Delta\rho_{-1\frac{1}{2}}$		$\Delta^3\rho_{-1\frac{1}{2}}$			
-1	ρ_{-1}		$\Delta^2\rho_{-1}$		$\Delta^4\rho_{-1}$		
		$\Delta\rho_{-\frac{1}{2}}$		$\Delta^3\rho_{-\frac{1}{2}}$		$\Delta^5\rho_{+\frac{1}{2}}$	
0	ρ_0		$\Delta^2\rho_0$		$\Delta^4\rho_0$		$\Delta^6\rho_0$
		$\Delta\rho_{+\frac{1}{2}}$		$\Delta^3\rho_{+\frac{1}{2}}$		$\Delta^5\rho_{+\frac{1}{2}}$	
+1	ρ_{+1}		$\Delta^2\rho_{+1}$		$\Delta^4\rho_{+1}$		
		$\Delta\rho_{+1\frac{1}{2}}$		$\Delta^3\rho_{+1\frac{1}{2}}$			
+2	ρ_{+2}		$\Delta^2\rho_{+2}$				
		$\Delta\rho_{+2\frac{1}{2}}$					
+3	ρ_{+3}						

The function

$$\rho_p(n) = \rho_0 + \Delta\rho_0 n + \Delta^2\rho_0 \frac{n^2}{2} + \Delta^3\rho_0 \frac{n(n^2-1)}{3!} + \Delta^4\rho_0 \frac{n^2(n^2-1)}{4!}$$

(6)

$$+ \Delta^5\rho_0 \frac{n(n^2-1)(n^2-2^2)}{5!} + \ldots + \Delta^{2p}\rho_0 \frac{n^2(n^2-1)(n^2-2^2)\ldots(n^2-(p-1)^2)}{(2p)!}$$

is obviously a polynomial in n and hence in θ. Also, it takes on the $2p + 1$ tabulated values of ρ from $n = -p$ to $n = +p$. The function is therefore a polynomial with which one can carry out the interpolation and differentiation.

The derivatives are

$$\frac{d\rho_p}{dn} = \Delta\rho_0 + \Delta^2\rho_0 n + \Delta^3\rho_0 \frac{3n^2-1}{3!} + \Delta^4\rho_0 \frac{4n^3-2n}{4!} + \Delta^5\rho_0 \frac{5n^4-15n^2+4}{5!} + \ldots,$$

$$\frac{d^2\rho_p}{dn^2} = \Delta^2\rho_0 + \Delta^3\rho_0 n + \Delta^4\rho_0 \frac{12n^2-2}{4!} + \Delta^5\rho_0 \frac{20n^3-30n}{5!} + \ldots .$$

At $n = 0$,

$$\left[\frac{d\rho_p}{dn}\right]_0 = \Delta\rho_0 - \frac{1}{6}\Delta^3\rho_0 + \frac{1}{30}\Delta^5\rho_0 + \ldots,$$

$$\left[\frac{d^2\rho_p}{dn^2}\right]_0 = \Delta^2\rho_0 - \frac{1}{12}\Delta^4\rho_0 + \ldots, \tag{7}$$

$$\left[\frac{d\rho_p}{dn}\right]_{\frac{1}{2}} = \Delta\rho_{\frac{1}{2}} - \frac{1}{24}\Delta^3\rho_{\frac{1}{2}} + \frac{3}{640}\Delta^5\rho_{\frac{1}{2}} + \ldots.$$

The derivatives are very nearly equal to the corresponding tabular differences when the higher differences are small. For example, at n = 0 the first difference in Table 1 is 4759.5, and the relative error introduced by using it as the first derivative is $\frac{4}{4763.5} = 0.0008$. The relative error introduced by using 361 for the second derivative is $\frac{0.5}{361.5} = 0.001$. This last formula, for calculating the derivative half way between the tabular entries, is given because of its greater convenience and more rapid convergence. It can, however, be derived in the same general fashion as the first equation.

If the higher differences do not become small, Equation (7) can be used to calculate the correct derivative. However, tabular differences that eventually increase indicate that the table may contain errors. Table 3 shows how errors in a tabular value introduce larger errors in the differences, and how the errors become larger for the higher differences. The round-off errors in a table eventually affect the differences sensibly, and the higher differences must be smoothed to remove these irregularities.

Differentiation of Equation (5) gives

$$\frac{dn}{d\theta} = \frac{1}{\Delta\theta}, \qquad \frac{d^2n}{d\theta^2} = 0.$$

The derivatives with respect to θ, which occur in the formulas for the radii of curvature and the distances, are

$$\rho'(\theta) = \frac{d\rho}{d\theta} = \frac{d\rho}{dn}\frac{dn}{d\theta} = \frac{1}{\Delta\theta}\frac{d\rho}{dn},$$

$$\rho''(\theta) = \frac{d^2\rho}{d\theta^2} = \frac{1}{\Delta\theta^2}\frac{d^2\rho}{dn^2}.$$

To calculate $\rho'(\theta)$ and $\rho''(\theta)$, the angle θ must be measured in radians; therefore $\Delta\theta$ must be the interval *in radians* between tabular entries.

Table 3 - Difference Table with Error

n	ρ	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
-3	ρ_{-3}						
		$\Delta\rho_{-2\frac{1}{2}}$					
-2	ρ_{-2}		$\Delta^2\rho_{-2}$				
		$\Delta\rho_{-1\frac{1}{2}}$		$\Delta^3\rho_{-1\frac{1}{2}}+\epsilon$			
-1	ρ_{-1}		$\Delta^2\rho_{-1}+\epsilon$		$\Delta^4\rho_{-1}-4\epsilon$		
		$\Delta\rho_{-\frac{1}{2}}+\epsilon$		$\Delta^3\rho_{-\frac{1}{2}}-3\epsilon$		$\Delta^5\rho_{-\frac{1}{2}}+10\epsilon$	
0	$\rho_0+\epsilon$		$\Delta^2\rho_0-2\epsilon$		$\Delta^4\rho_0+6\epsilon$		$\Delta^6\rho_0-20\epsilon$
		$\Delta\rho_{+\frac{1}{2}}-\epsilon$		$\Delta^3\rho_{+\frac{1}{2}}+3\epsilon$		$\Delta^5\rho_{+\frac{1}{2}}-10\epsilon$	
+1	ρ_{+1}		$\Delta^2\rho_{+1}+\epsilon$		$\Delta^4\rho_{+1}-4\epsilon$		
		$\Delta\rho_{+1\frac{1}{2}}$		$\Delta^3\rho_{+1\frac{1}{2}}-\epsilon$			
+2	ρ_{+2}		$\Delta^2\rho_{+2}$				
		$\Delta\rho_{+2\frac{1}{2}}$					
+3	ρ_{+3}						

In order that Equation (7) may be used to calculate the derivatives, it must be assumed that the cam shape can be approximated by a polynomial. When this assumption is not valid, the derivatives may be in error. For a cam composed of flat facets, the lift can be calculated directly for *all* cam angles by a consideration of the geometrical configuration. The intermediate values so calculated are different from the values obtained from polynomial interpolation, and this polynomial interpolation is not valid for such a cam. It is not immediately obvious that the cam, as it wears into a smooth curve, comes any closer to a shape for which polynomial interpolation can be used.

Calculation of $\int\rho(\theta)\,d\theta$. If the function $\rho(\theta)$ is defined by a table, it must be extended to the intervals between tabular entries before the integral can be calculated. However, the value of the integral is less sensitive to the kind of interpolation used than are the derivatives, because integration is a smoothing process whereas differentiation is not. Generally polynomial interpolation is adequate, the only exception being the case in which the interpolated values of $\rho(\theta)$ ought to be systematically above (or below) a smooth curve through the tabular values.

The polynomial ρ_p (n) given by Equation (6) is a suitable interpolation function. Its integral is

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \rho_p\,(n)\,dn = \left[\rho_0\, n + \Delta\,\rho_0\,\frac{n^2}{2} + \Delta^2\rho_0\,\frac{n^3}{3!} + \Delta^3\rho_0\,\frac{n^4-2n^2}{4!} + \Delta^4\rho_0\,\frac{3n^5-5n^3}{5!3} + \ldots \right]_{-\frac{1}{2}}^{+\frac{1}{2}} = \rho_0 + \frac{1}{24}\,\Delta^2\rho_0 - \frac{17}{5760}\,\Delta^4\rho_0 + \ldots .$$

The integral is more conveniently calculated for values of n, or θ, that are half way between the values for which $\rho(\theta)$ is tabulated. The increment between two successive values of the integral is very nearly the intervening value of $\rho(\theta)$, and one can calculate the integral by adding the tabulated values of $\rho(\theta)$, correcting the sum, if necessary, with the formula above. For the function shown in Table 1, the increase in the integral from n = -1/2 to n = +1/2 is

$$35725 + \frac{361}{24} + \frac{17x6}{5760} = 35740.$$

The term $\int\rho(\theta)\,d\theta$ is required in the formula for the distance along the cam surface. From Equation (5),

$$d\theta = \Delta\theta\; dn$$

and

$$\int\rho(\theta)\,d\theta = \Delta\theta \int\rho_p\,(n)\,dn.$$

Once again, θ must be in radians. The value of $\int\rho(\theta)\,d\theta$ for each θ is obtained by adding the corresponding $\Delta\theta \int_{-\frac{1}{2}}^{+\frac{1}{2}} \rho_p\,(n)\,dn$ to $\int\rho(\theta)\,d\theta$ for the previous θ.

The constant of integration is selected such that $\int\rho(\theta)\,d\theta$ is zero for $\theta = 0$. It is necessary to evaluate $\int\rho(\theta)\,d\theta$ at some point midway between the tabulated values of θ in order to start the iterative process for the calculation of subsequent values of $\int\rho(\theta)\,d\theta$. From Equation (6),

$$\int_0^{\epsilon\Delta\theta}\rho(\theta)\,d\theta = \Delta\theta\int_0^{\epsilon}\rho_p\,(n)\,dn = \Delta\theta\left[\rho_0\,\epsilon + \Delta\rho_0\,\frac{\epsilon^2}{2} + \Delta^2\rho_0\,\frac{\epsilon^3}{3!} + \Delta^3\rho_0\,\frac{\epsilon^4-2\epsilon^2}{4!} + \ldots\right],$$

in which ϵ is the fraction of the interval length $\Delta\theta$ between zero and the next center of an integral. If $\theta = 0$ is a tabular value, then $\epsilon = 1/2$ and the integral is

$$\int_0^{\frac{1}{2}\Delta\theta} \rho(\theta)\, d\theta = \Delta\theta \left[\frac{1}{2}\rho_0 + \frac{1}{8}\Delta\rho_0 + \frac{1}{48}\Delta^2\rho_0 - \frac{7}{384}\Delta^3\rho_0 + \dots \right].$$

At $\theta = 0$, $\rho(\theta)$ is a maximum and $\Delta\rho_0$ is quite small.

Summary. In calculating the stresses in a cam follower, one must know the radius of curvature of the cam and the acceleration of the follower. For a flat follower they are

$$\text{radius of curvature} = \frac{d^2\rho}{d\theta^2} + \rho(\theta),$$

$$\text{acceleration} = \frac{d^2\rho}{d\theta^2}\left[\frac{d\theta}{dt}\right]^2,$$

in which ρ is the lift and θ is the angle of rotation of the cam *in radians*.

When examining the areas of high stress in the cam and follower, it is convenient to know the distances along their surfaces to the point of contact. For a flat follower they are

$$\text{distance along follower} = \frac{d\rho}{d\theta},$$

$$\text{distance along cam} = \frac{d\rho}{d\theta} + \int_0^{\theta} \rho(\theta)\, d\theta,$$

measured from the points in contact at maximum lift.

In case $\rho(\theta)$ is defined by a table, and polynomial interpolation is permissible,

$$\int_0^{\theta} \rho(\theta)\, d\theta = \int_0^{\frac{1}{2}\Delta\theta} \rho(\theta)\, d\theta + \Delta\theta \left[\Sigma\rho + \frac{1}{24}\Sigma\Delta^2\rho - \frac{17}{5760}\Sigma\Delta^4\rho + \dots \right],$$

$$\frac{d\rho}{d\theta} = \frac{1}{\Delta\theta}\left[\Delta\rho - \frac{1}{24}\Delta^3\rho + \frac{3}{640}\Delta^5\rho + \dots \right],$$

$$\frac{d^2\rho}{d\theta^2} = \frac{1}{\Delta\theta^2}\left[\Delta^2\rho - \frac{1}{12}\Delta^4\rho + \dots \right].$$

The integral and first derivative are obtained for values half way between the tabular entries, and the second derivative is evaluated at the entries ($\theta = 0$ is an entry). The distance between entries is $\Delta\theta$, *in radians*. The integral $\int_0^{\frac{1}{2}\Delta\theta} \rho(\theta)\, d\theta$ is very nearly $\frac{1}{2}\rho(\theta)$, and the summations are over the values after $\theta = 0$.

6

The Effect of Inclusions on the Fatigue Life of Steel

By Leonard G. Johnson
Research Laboratories Division
General Motors Corporation

INTRODUCTION

It is generally admitted that non-metallic inclusions and other flaws in steel act as nuclei of stress concentration around which early fatigue failures can originate. This being so, it is important to be able to estimate what reduction in fatigue life can be expected for a given inclusion content which has some representative stress concentration factor. By treating each volume element in a material as an independent component having a Weibull distribution of life, dependent upon its particular stress level, it is possible to make a theoretical prediction of the fatigue life for any specimen consisting of any number of such volume elements simply by multiplying together the independent probabilities of failure of the separate elements. In order to predict the effect of inclusions on fatigue life we need to know how many volume elements contain inclusions, and what the stresses are both in the elements containing inclusions and in those elements not containing inclusions. By means of these basic concepts a mathematical model of the effect of inclusions on fatigue life can be constructed. Such a theoretical guide has been needed for a long time, and its construction will now be demonstrated.

NOTATION

v = Total stressed volume under consideration (grid of Fig. 1) = 1 (consisting of k equally sized elements or components)

k = Total number of components of volume under consideration

V_i = Volume of the i^{th} component = $\frac{v}{k} = \frac{1}{k}$

s = The nominal stress in the total volume under consideration

s_i = The stress in the i^{th} component = $\lambda_i s$

λ_i = The stress concentration factor in the i^{th} component

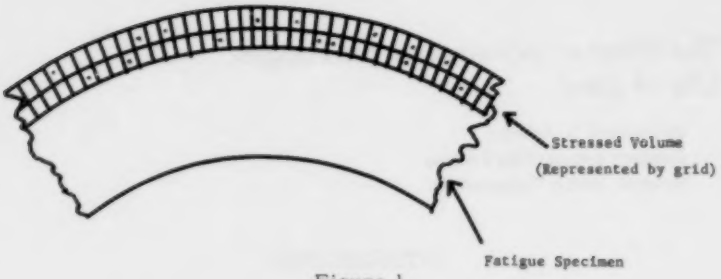


Figure 1.

c = The number of components of volume which contain inclusions (those with dots in Figure 1)

v_c = The sum of the volumes of all components having inclusions

$f_c = \frac{v_c}{v} = \frac{c}{k}$ = fraction of the total stressed volume having inclusions

POSTULATES OF THE MATHEMATICAL MODEL

Postulate 1: The i^{th} component (element of volume) has a life distribution defined by the Weibull survivorship function [1]*

$$p_i(x) = \exp\left[-\left(\frac{x}{n_i}\right)^b\right] \tag{1}$$

where b is a positive constant having the same value for all components (known as the Weibull slope),

and where n_i = characteristic life of the i^{th} component, i.e., the life for which the survival probability is $e^{-1} = .368$.

($p_i(x)$ denotes the probability that the i^{th} component will survive to a life of x.)

Postulate 2: The relationship between the characteristic life of the i^{th} component and the stress on the i^{th} component is given by a formula of the type

$$n_i = \frac{1}{P(s_i)\, V_i^{\frac{1}{b}}} = \frac{k^{\frac{1}{b}}}{P(s_i)} \tag{2}$$

*Numbers in square brackets refer to references at the end of the paper.

where $P(s_i)$ is a monotone increasing function of s_i (known as the potency of s_i) (see [2])

(For pitting fatigue in bearnings, $P(s_i)$ is generally taken to be a simple power function of s_i, i.e., $P(s_i)$ = constant x s_i^m ($m > 0$))

Postulate 3: The components (elements of volume) comprising the specimen are statistically independent.

DERIVATION OF THE LIFE PREDICTION EQUATION

According to Postulate 1:

$$p_i(x) = \exp\left[-(x/n_i)^b \right] \tag{3}$$

Since according to Postulate 3 the components are statistically independent we can arrive at the survival probability of the entire specimen by using *the product rule* [3] for independent survival probabilities. Thus,

If p(x) = the probability that the entire assembly (specimen) survives to life x

It follows that

$$p(x) = p_1(x)\, p_2(x)\, p_3(x) \cdots\cdots p_k(x) \tag{4}$$

(k components)

$$\text{or} \quad p(x) = \exp\left[-x^b (1/n_1^b + 1/n_2^b + 1/n_3^b + \ldots + 1/n_k^b) \right] \tag{5}$$

Let $\bar{n}$ = the characteristic life of the assembly (specimen)

Then $$p(x) = \exp\left[-x^b/\bar{n}^b \right] \tag{6}$$

$$\therefore\ 1/\bar{n}^b = 1/n_1^b + 1/n_2^b + 1/n_3^b + \ldots\ldots + 1/n_k^b \tag{7}$$

Let us denote by subscripts 1,2,3, ... ,c those c components having inclusions.

Let $n_1, n_2, n_3, \ldots, n_c$ be the respective characteristic lives of these c components.

Let n = characteristic life of each component without inclusions (each at the same nominal stress s)
(There are k-c such inclusion-free components)

Then (7) becomes

$$1/\bar{n}^b = 1/n_1^b + 1/n_2^b + \ldots + 1/n_c^b + (k-c)/n^b \tag{8}$$

But, according to (2)

$$1/n_i^b = \frac{[P(s_i)]^b}{k} \quad \text{and} \quad 1/n_i^b = \frac{[P(s)]^b}{k}$$

$$\therefore\ 1/\bar{n}^b = \frac{[P(s_1)]^b + [P(s_2)]^b + \ldots\ldots + [P(s_c)]^b}{k} + \frac{(k-c)[P(s)]^b}{k}$$

$$\text{or}\ 1/\bar{n}^b = \frac{[P(\lambda_1 s)]^b + [P(\lambda_2 s)]^b + \ldots\ldots + [P(\lambda_c s)]^b}{k} + \frac{(k-c)[P(s)]^b}{k}$$

Let $\hat{\lambda}$ = an "average" stress concentration factor such that

$$[P(\lambda_1 s)]^b + [P(\lambda_2 s)]^b + \ldots\ldots + [P(\lambda_c s)]^b = c\,[P(\hat{\lambda} s)]^b$$

Then

$$1/\bar{n}^b = (c/k)\,[P(\hat{\lambda} s)]^b + (1 - c/k)\,[P(s)]^b \qquad (9)$$

or,

$$1/\bar{n}^b = f_c[P(\hat{\lambda} s)]^b + (1 - f_c)\,[P(s)]^b \qquad (10)$$

For the entire assembly (having unit stressed volume) we have (see Postulate 2)

$$P(s) = 1/n_s \quad \left(n_s = \text{characteristic life of assembly at stress } s\right)$$

$$\text{and}\quad P(\hat{\lambda} s) = 1/n_{\hat{\lambda} s} \quad \left(n_{\hat{\lambda} s} = \text{characteristic life of assembly at stress } \hat{\lambda} s\right)$$

Thus, (10) reduces to

$$1/\bar{n}^b = f_c/n_{\hat{\lambda} s}^b + \frac{1 - f_c}{n_s^b} \qquad (11)$$

Equation (11) represents the fundamental life prediction equation for a material having the fraction f_c of its total volume consisting of inclusions with a stress concentration factor $\hat{\lambda}$.

APPLICATION OF THE FUNDAMENTAL LIFE PREDICTION EQUATION

Let us illustrate the use of the fundamental equation (11) by applying it to a bearing problem where the s--n relationship is

$$n_s = 1/s^m \qquad (m > 0) \qquad (12)$$

$$\therefore \quad n_{\hat{\lambda}_s} = 1/(\hat{\lambda}^m \, s^m) \tag{13}$$

$\therefore$ (11) becomes

$$1/\bar{n}^b = f_c \, \hat{\lambda}^{mb} \, s^{mb} + (1 - f_c) \, s^{mb} = (1 - f_c + f_c \, \hat{\lambda}^{mb}) \, s^{mb}$$

$$\text{or} \quad \bar{n}^b = \frac{1}{(1 - f_c + f_c \, \hat{\lambda}^{mb}) \, s^{mb}} = \frac{n_s^b}{1 - f_c + f_c \, \hat{\lambda}^{mb}}$$

$$\text{or} \quad \bar{n} = \frac{n_s}{(1 - f_c + f_c \, \hat{\lambda}^{mb})^{\frac{1}{b}}}$$

$$\therefore \quad \bar{n}/n_s = \frac{1}{(1 - f_c + f_c \, \hat{\lambda}^{mb})^{\frac{1}{b}}} \tag{14}$$

Equation (14) gives us the ratio of the specimen life when the fraction f_c of the volume consists of inclusions of stress concentration factor $\hat{\lambda}$ to the life of an inclusion-free specimen.

NUMERICAL EXAMPLE

If the Weibull slope is $b = 1$, and if $m = 12$, it follows that if 1/1000 of the volume of bearing material consists of inclusions of stress concentration factor $\hat{\lambda} = 2$ we can expect a fatigue life

$$\bar{n} = \frac{n_s}{1 - .001 + .001 \, (2^{12})} = .19627 \, n_s$$

In other words, due to inclusions of stress concentration factor 2 in 1/1000 of the volume, the life is reduced to less than 1/5 of what it would be for an inclusion-free bearing at the same nominal stress level.

(The results for different stress concentration factors are shown graphically in Figure 2.)

CONCLUSION

By assuming Weibull life distributions of fixed slope and statistically independent volume elements we have derived a simple life prediction equation which involves

1. The fraction f_c by volume of material which contains inclusion.
2. The stress concentration factor $\hat{\lambda}$ of the inclusions.

But, according to (2)

$$1/n_i^b = \frac{[P(s_i)]^b}{k} \quad \text{and} \quad 1/n_1^b = \frac{[P(s)]^b}{k}$$

$$\therefore\ 1/\bar{n}^b = \frac{[P(s_1)]^b + [P(s_2)]^b + \ldots\ldots + [P(s_c)]^b}{k} + \frac{(k-c)[P(s)]^b}{k}$$

$$\text{or } 1/\bar{n}^b = \frac{[P(\lambda_1 s)]^b + [P(\lambda_2 s)]^b + \ldots\ldots + [P(\lambda_c s)]^b}{k} + \frac{(k-c)[P(s)]^b}{k}$$

Let $\hat{\lambda}$ = an "average" stress concentration factor such that

$$[P(\lambda_1 s)]^b + [P(\lambda_2 s)]^b + \ldots\ldots + [P(\lambda_c s)]^b = c\,[P(\hat{\lambda} s)]^b$$

Then

$$1/\bar{n}^b = (c/k)\,[P(\hat{\lambda} s)]^b + (1 - c/k)\,[P(s)]^b \qquad (9)$$

or,

$$1/\bar{n}^b = f_c[P(\hat{\lambda} s)]^b + (1 - f_c)\,[P(s)]^b \qquad (10)$$

For the entire assembly (having unit stressed volume) we have (see Postulate 2)

$$P(s) = 1/n_s \quad \left(n_s = \text{characteristic life of assembly at stress } s\right)$$

$$\text{and} \quad P(\hat{\lambda} s) = 1/n_{\hat{\lambda} s} \quad \left(n_{\hat{\lambda} s} = \text{characteristic life of assembly at stress } \hat{\lambda} s\right)$$

Thus, (10) reduces to

$$1/\bar{n}^b = f_c/n_{\hat{\lambda} s}^b + \frac{1 - f_c}{n_s^b} \qquad (11)$$

Equation (11) represents the fundamental life prediction equation for a material having the fraction f_c of its total volume consisting of inclusions with a stress concentration factor $\hat{\lambda}$.

APPLICATION OF THE FUNDAMENTAL LIFE PREDICTION EQUATION

Let us illustrate the use of the fundamental equation (11) by applying it to a bearing problem where the s--n relationship is

$$n_s = 1/s^m \qquad (m > 0) \qquad (12)$$

$$\therefore \quad n_{\hat{\lambda}_s} = 1/(\hat{\lambda}^m s^m) \tag{13}$$

$\therefore$ (11) becomes

$$1/\bar{n}^b = f_c \hat{\lambda}^{mb} s^{mb} + (1 - f_c) s^{mb} = (1 - f_c + f_c \hat{\lambda}^{mb}) s^{mb}$$

$$\text{or} \quad \bar{n}^b = \frac{1}{(1 - f_c + f_c \hat{\lambda}^{mb}) s^{mb}} = \frac{n_s^b}{1 - f_c + f_c \hat{\lambda}^{mb}}$$

$$\text{or} \quad \bar{n} = \frac{n_s}{(1 - f_c + f_c \hat{\lambda}^{mb})^{\frac{1}{b}}}$$

$$\therefore \quad \bar{n}/n_s = \frac{1}{(1 - f_c + f_c \hat{\lambda}^{mb})^{\frac{1}{b}}} \tag{14}$$

Equation (14) gives us the ratio of the specimen life when the fraction f_c of the volume consists of inclusions of stress concentration factor $\hat{\lambda}$ to the life of an inclusion-free specimen.

NUMERICAL EXAMPLE

If the Weibull slope is b = 1, and if m = 12, it follows that if 1/1000 of the volume of bearing material consists of inclusions of stress concentration factor $\hat{\lambda} = 2$ we can expect a fatigue life

$$\bar{n} = \frac{n_s}{1 - .001 + .001\ (2^{12})} = .19627\ n_s$$

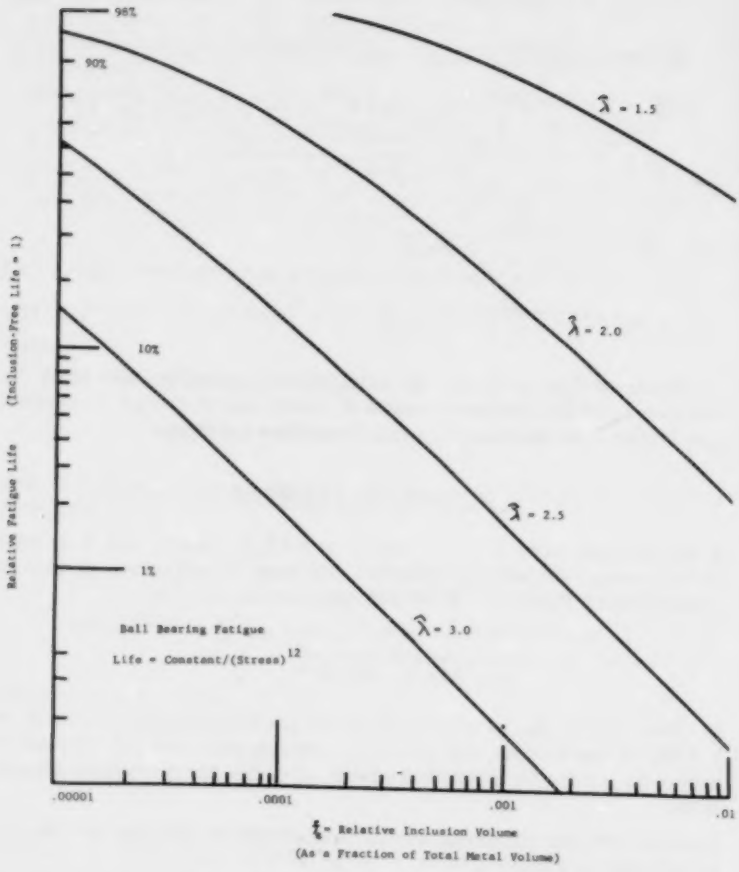
In other words, due to inclusions of stress concentration factor 2 in 1/1000 of the volume, the life is reduced to less than 1/5 of what it would be for an inclusion-free bearing at the same nominal stress level.

(The results for different stress concentration factors are shown graphically in Figure 2.)

CONCLUSION

By assuming Weibull life distributions of fixed slope and statistically independent volume elements we have derived a simple life prediction equation which involves

1. The fraction f_c by volume of material which contains inclusion.
2. The stress concentration factor $\hat{\lambda}$ of the inclusions.



Fatigue Life vs Inclusion Volume at Weibull Slope 1.0

Figure 2.

Application and verification of this predicting equation requires that methods be devised whereby inclusion volumes and stress concentration factors due to inclusions can be accurately determined.

REFERENCES

1. Johnson, L. G. : The Statistical Treatment of Fatigue Experiments (page 87)
Research Laboratories, General Motors Corp.
Report GMR-202

2. Johnson, L. G. : GMR Reliability Manual (page 6-6)
Research Laboratories, General Motors Corp.
Report GMR-302

3. Koff, R. M. : What the Experts Tell Us About Mechanical Reliability
Product Engineering (Nov. 16, 1959)
Page 23 (combination effects)

INDUSTRIAL MATHEMATICS is published semi-annually by the Industrial Mathematics Society. The advance subscription price is $4.00 per annual volume of two issues. Back issues (one volume per year) are available at $3.00 per copy. Orders accompanied by check or money order should be addressed to Industrial Mathematics Society, 100 Farnsworth Street, Detroit 2, Michigan.

INDUSTRIAL MATHEMATICS contains original articles as well as expository articles dealing with the application of mathematics in industry. Persons having such manuscripts for publication should submit them in duplicate to the Editor, INDUSTRIAL MATHEMATICS, 100 Farnsworth Street, Detroit 2, Michigan. Any questions on editorial matters should be similarly addressed.

Manuscripts should be typed double-spaced on one side of $8\frac{1}{2}$ by 11 inch paper in complete and finished form precisely as they are to appear, and sent – packed flat – to the Editor. Where possible, Figures should be numbered consecutively in arabic numerals. Tables should be numbered in arabic numerals, and both figures and tables should have captions.

Authors will receive galley proofs for correction purposes. One-hundred reprints without covers will be furnished free.

Author's institutions or companies are asked to pay a publication charge of $7.50 per printed page for articles accepted for publication.